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Dialysis with ultrafiltration through countercurrently parallel-flow membrane modules

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Abstract. The application of ultrafiltration operation to the dialysis in countercurrently parallel-flow rectangular membrane modules was investigated. The assumption of uniform ultrafiltration flux was made for operation with slight concentration polarization and declination of transmembrane pressure. Considerable improvement in mass transfer is achievable if the operation of ultrafiltration is applied, especially for the system with low mass transfer coefficient. The enhancement in separation efficiency is significantly increased with increasing ultrafiltration flux, as well as with increasing the volumetric flow rates. Furthermore, increasing the volumetric flow rate in retentate phase is more beneficial to mass transfer than increasing in dialysate phase.

Keywords: mass transfer; dialysis; ultrafiltration; countercurrently parallel-flow; rectangular membrane module

1. Introduction

The famous applications of dialysis are hemodialysis for removing the metabolic waste from the blood and the recovery of acids from various waste solutions by employing ion exchange membranes (Kobuchi *et al*. 1986, Narebska and Warszaski 1994, Palaty and Zakova 1996, Palaty *et al*. 2000, Oh *et al*. 2000). Grimsurd and Babb (1966) and Cooney *et al*. (1974) analyzed the effect of dialysis in plate-type and tubular-type membranes, respectively, with the considerations that the flow is fully developed and that the fluid is incompressible. They also assumed that the concentration of solute in the dialysate is constant.

Intradialytic hypotension remains the most frequent hemodialysis complication, occurring in approximately 25% of dialysis sessions (Palmer and Henrich 2008) and may be an independent predictor of cardiovascular mortality in this patient group. (Shoji *et al*. 2004, Tisler *et al*. 2003). The first line of treatment for symptomatic intradialytic hypotension is pausing ultrafiltration (Davenport 2009) placing the patient into a reclined position, with possible administration of normal saline (Knoll *et al*. 2004). Later Brdshaw (2010) applied pre-emptively ultrafiltration to minimize dialysis hypotension.

An effective approach to enhance the removal of larger substances is to increase the rate of fluid filtration so that the substances can be carried through the membrane by convective forces.

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Popovich *et al*. (1971) considered both effects of dialysis and constant ultrafiltration flux in plate-type membrane modules, and assumed a zero solute concentration in the dialysate phase. Jagannathan and Shettigar (1977) discussed both effects in hollow fiber membrane modules with the considerations of a variable dialysate concentration and a constant ultrafiltration flux along the length of the dialyzer. The results show that the clearances of the solute are affected significantly by the ultrafiltration flux, solute permeability of the membrane, and concentration of the dialysate. The mass transfers for dialysis coupled with uniform and nonuniform ultrafiltration fluxes in cross-flow membrane module were analyzed by the methods of perturbation technique and orthogonal collocation, respectively (Yeh *et al*. 1997, 2000). Recently, several devices of ultrafiltration (Bourge and Tallaj 2005, Costanzo *et al*. 2005, 2007, Kazory and Ross 2008) and peritoneal dialysis (Nakayama *et al*. 2010) for severe heart failure have been introduced. It is the purpose of present study to analyze the mass transfer for dialysis coupled with uniform ultrafiltration flux in countercurrently parallel-flow rectangular membrane modules. The results will be compared with those obtained in the device without ultrafiltration.

2. Theory

In this study, we will deal with the dialysis coupled with uniform ultrafiltration flux in the countercurrently parallel-flow rectangular membrane modules. The assumptions made in this analysis are: steady state, no chemical reaction, uniform concentrations and velocities over the cross section of flow, constant mass transfer coefficients and uniform ultrafiltration flux, V_m, for the operation with the neglect of the concentration polarization and the declination of transmembrane pressure.

2.1 Concentration distributions

The schematic diagram in Fig. 1 may serve to explain the nomenclature to be employed for countercurrent-flow operation. The system consists of two channels, for the fluids B (retentate phase) and D (dialysate phase), respectively, separated by a microporous membrane sheet which solute is transferred from the retentate side to the dialysate side.

The volumetric balances for the solutions in a differential length dx are

$$-\frac{dQ_B}{Wdx} = V_m \tag{1}$$

$$-\frac{dQ_D}{Wdx} = V_m \tag{2}$$

where $Q_B(x)$ and $Q_D(x)$ denote the volumetric flow rates of retentate and dialysate phases, respectively, x is the coordinate along dialyzer, and W and L are the width and length of flow channels. Integrating Eqs. (1) and (2) with the initial conditions: $Q_B = B_{B,i}$ at $x = 0$ and $Q_D = Q_{D,i}$ at $x = L$, respectively, one has

$$Q_B = Q_{B,i} - WV_m x \tag{3}$$

$$Q_D = Q_{D,i} - WV_m(x - L) \tag{4}$$

and their average values along the whole length of channels L are

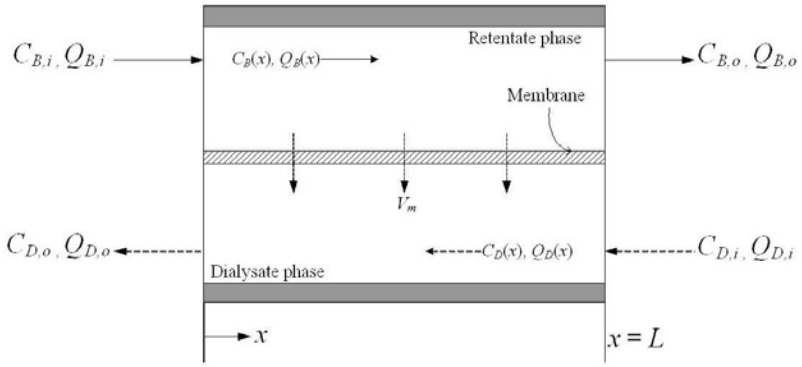

Fig. 1 Flows and fluxes in a countercurrent-flow rectangular dialyzer with ultrafiltration

$$Q_{B,m} = \left(\frac{1}{L}\right)\int_0^L Q_B dx = Q_{B,i} - \left(\frac{WLV_m}{2}\right) \tag{5}$$

$$Q_{D,m} = \left(\frac{1}{L}\right)\int_0^L Q_D dx = Q_{D,i} + \left(\frac{WLV_m}{2}\right) \tag{6}$$

Taking the differential mass balances for solute in both channels, we have

$$-\frac{d(Q_B C_B)}{Wdx} = K(C_B - C_D) + V_m \theta C_B \tag{7}$$

$$-\frac{d(Q_D C_D)}{Wdx} = K(C_B - C_D) + V_m \theta C_B \tag{8}$$

where K is the overall mass transfer coefficient of the solute and θ is the membrane sieving coefficient for solute, where $\theta = 1$ for not rejected by the membrane, $\theta < 1$ for partially rejected, and $\theta = 0$ for completely rejected. If the ultrafiltration rate is very small compared with the volumetric flow rates, i.e., $V_m LW << Q_{D,i}$, we may assume that $Q_B \approx Q_{B,m}$ and $Q_D \approx Q_{D,m}$ in Eqs. (7) and (8). Therefore, Eqs. (7) and (8) can be rewritten with the use of Eqs. (5) and (6), as

$$-\left[\left(\frac{1}{a}\right) - \left(\frac{\phi}{2}\right)\right]\frac{d\varsigma_B}{d\xi} = (\varsigma_B - \varsigma_D) + \phi\theta\varsigma_B \tag{9}$$

$$-\left[\left(\frac{1}{b}\right) + \left(\frac{\phi}{2}\right)\right]\frac{d\varsigma_D}{d\xi} = (\varsigma_B - \varsigma_D) + \phi\theta\varsigma_B \tag{10}$$

where

$$\varsigma_B = \frac{C_B}{C_{B,i}} \tag{11}$$

$$\varsigma_D = \frac{C_D}{C_{B,i}} \tag{12}$$

$$\xi = \frac{x}{L} \tag{13}$$

$$a = \frac{LWK}{Q_{B,i}} \tag{14}$$

$$b = \frac{LWK}{Q_{D,i}} \tag{15}$$

$$\phi = \frac{V_m}{K} \tag{16}$$

Eq. (9) may be rewritten as

$$\varsigma_D = (\phi\theta + 1)\varsigma_B + \left[\left(\frac{1}{a}\right) - \left(\frac{\phi}{2}\right)\right]\frac{d\varsigma_B}{d\xi} \tag{17}$$

Differentiation of Eq. (17) yields

$$\frac{d\varsigma_D}{d\xi} = (\phi\theta + 1)\frac{d\varsigma_B}{d\xi} + \left[\left(\frac{1}{a}\right) - \left(\frac{\phi}{2}\right)\right]\frac{d^2\varsigma_B}{d\xi^2} \tag{18}$$

Substitution of Eqs. (17) and (18) into Eq. (10) and rearrangement results in

$$\frac{d^2\varsigma_B}{d\xi^2} + A\frac{d\varsigma_B}{d\xi} = 0 \tag{19}$$

where

$$A = \frac{(\phi\theta + 1)}{[(1/a) - (\phi/2)]} - \frac{1}{[(1/b) + (\phi/2)]} \tag{20}$$

Integrating Eq. (19) twice, one obtains

$$\frac{d\varsigma_B}{d\xi} = me^{-A\xi} \tag{21}$$

$$\varsigma_B = -\left(\frac{m}{A}\right)e^{-A\xi} + n \tag{22}$$

where m and n are two integration constants to be determined. Substitution of Eqs. (21) and (22) into Eq. (17) gives

$$\varsigma_D = -mBe^{-A\xi} + Dn \tag{23}$$

where

$$B = \left(\frac{D}{A}\right) - \left(\frac{1}{a}\right) + \left(\frac{\phi}{2}\right) \tag{24}$$

$$D = \phi\theta + 1 \tag{25}$$

By applying the inlet solute concentrations: $\varsigma_B = 1$ at $\xi = 0$ and $\varsigma_D = \varepsilon$ at $\xi = 1$, one has

$$1 = -m\left(\frac{1}{A}\right) + n \tag{26}$$

$$\varepsilon = -mBe^{-A} + nD \tag{27}$$

where

$$\varepsilon = \frac{C_{D,i}}{C_{B,i}} \tag{28}$$

Solving Eqs. (26) and (27) for m and n, we obtain

$$m = \frac{-(D-\varepsilon)}{(D/A) - Be^{-A}} \tag{29}$$

$$n = \frac{\left[-(D-\varepsilon)/A\right]}{(D/A) - Be^{-A}} + 1 \tag{30}$$

Finally, the outlet solute concentrations are readily obtained by applying the boundary conditions: $\varsigma_B = \varsigma_{B,o} = (C_{B,o} / C_{B,i})$ at $\xi = 1$ and $\varsigma_D = \varsigma_{D,o} = (C_{D,o} / C_{B,i})$ at $\xi = 0$. The results are

$$\varsigma_{B,o} = -m\left(\frac{1}{A}\right)e^{-A} + n \tag{31}$$

$$\varsigma_{D,o} = -mB + Dn \tag{32}$$

2.2 Mass transfer rate

Once one of the outlet solute concentrations is obtained, the total mass transfer rate can be calculated from the overall mass balance as

$$M = Q_{B,i}C_{B,i} - Q_{B,o}C_{B,o} \tag{33}$$

The outlet volumetric flow rate of the retentate phase may be obtained from Eq. (3) by setting $x = L$, i.e.

$$Q_{B,o} = Q_{B,i} - WV_mL = Q_{B,i}(1 - a\phi) \tag{34}$$

Eq. (33) becomes

$$M = Q_{B,i}C_{B,i}\left[1 - (1 - a\phi)\varsigma_{B,o}\right] \tag{35}$$

Table 1 Improvement of performance in the device with $C_{B,i}$ = 0.5 kg/m³, $C_{D,i}$ = 0, $Q_{D,i}$ = 4 × 10^{-6} m³/s

$Q_{B,i} \times 10^6$ (m^3/s)	$V_m LW \times 10^6$ (m^3/s)	$C_{B,o}$ (kg/m³)		$M \times 10^6$ (kg/s)		I (%)	
		$\theta = 1$	$\theta = 0.61$	$\theta = 1$	$\theta = 0.61$	$\theta = 1$	$\theta = 0.61$
2	0	0.064	0.343	0.871	0.314	0	0
2	0.05	0.061	0.337	0.881	0.344	1.176	9.556
2	0.1	0.057	0.330	0.891	0.373	2.288	18.932
2	0.2	0.051	0.317	0.909	0.430	4.327	37.130
4	0	0.202	0.415	1.192	0.341	0	0
4	0.05	0.198	0.411	1.217	0.376	2.050	10.053
4	0.1	0.195	0.408	1.241	0.410	4.067	20.027
4	0.2	0.187	0.401	1.288	0.477	8.002	39.733
4	0.4	0.173	0.387	1.377	0.608	15.470	78.152
8	0	0.329	0.455	1.372	0.356	0	0
8	0.05	0.326	0.454	1.406	0.393	2.466	10.281
8	0.1	0.324	0.452	1.439	0.429	4.919	20.527
8	0.2	0.320	0.448	1.506	0.502	9.787	40.912
8	0.4	0.311	0.441	1.638	0.645	19.367	81.245
8	0.8	0.293	0.427	1.891	0.926	37.864	160.093

Accordingly, the improvement in mass transfer rate by applying the uniform ultrafiltration flux may be designed based on that $M|_{V_m=0}$ obtained from the performance of pure dialysis without ultrafiltration, as

$$I = \frac{M - M|_{V_m=0}}{M|_{V_m=0}} \tag{36}$$

3. Numerical calculation

3.1 Numerical example

For the purpose of illustration, let us employ the experimental system (Kunitomo *et al*. 1978, Sakai and Mineshima 1984) for removing urea and inulin from blood by the filtryzer B-1-M. This membrane module has an effective membrane area LW of 1.36 m^2 and sieving coefficients of 1.0 and 0.61 for urea and inulin, respectively. The overall mass transfer coefficients for urea and inulin in the module system are 4.342 × 10^{-6} and 6.05 × 10^{-7} m/s, respectively.

3.2 Results and discussion

With the use of above numerical values, the solute outlet concentrations in retentate phase $\varsigma_{B,o}$

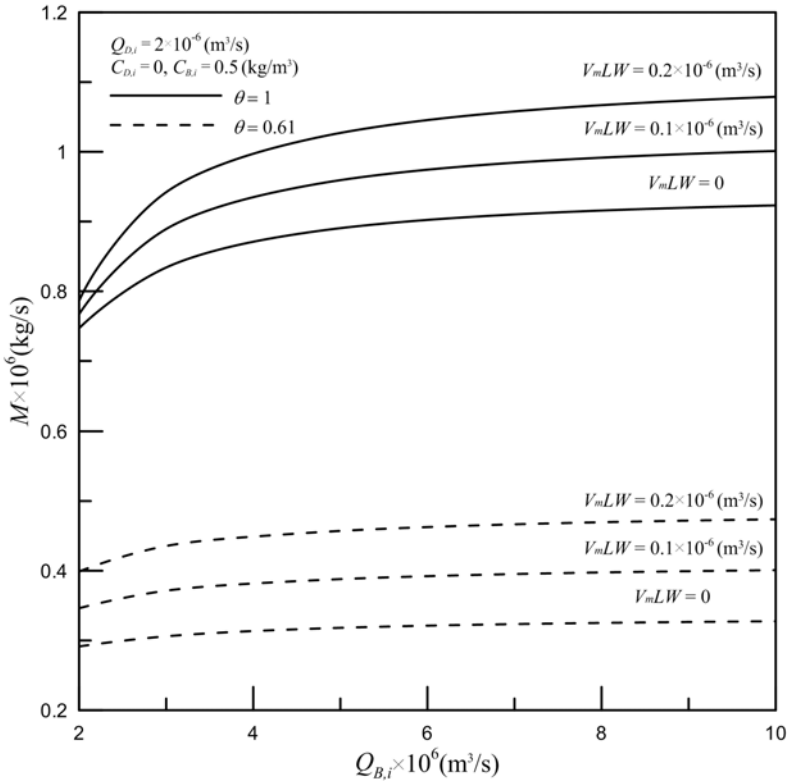

Fig. 2 Mass transfer rate for $Q_{D,i} = 2 \times 10^{-6}$ m³/s, $C_{B,i} = 0.5$ kg/m³, $C_{D,i} = 0$

are calculated from Eq. (31). Finally, the overall mass transfer rates M and the improvements in performance I by applying the ultrafiltration flux obtained from Eqs. (35) and (36), respectively. The results are presents in Table 1 for $Q_{D,i} = 4 \times 10^{-6}$ m³/s and in Figs. 2-5 for $Q_{D,i} = 2 \times 10^{-6}$ m³/s and 8×10^{-6} m³/s.

3.2.1 Outlet concentration

The results of solute outlet concentrations in retentate phase are given in Table 1 with various operating conditions. For the case of θ =1, urea is not rejected by the membrane, and the outlet concentration decreases when the retentate flow rate $Q_{B,i}$ decreases, as well as when the ultrafiltration rate (V_mLW) increases. On the other hand, when the retentate flow rate is larger, the residence time of solute in the module is shorter for solute to transfer through the membrane, and the outlet concentration approaches the inlet concentration. For the case of θ = 0.61, the solute is partially rejected by the membrane, while the solvent is removed freely by the effect of ultrafiltration, and the solute outlet concentration in retentate phase is higher than that in the case of θ = 1.

3.2.2 Mass transfer rate

After the solute outlet concentrations $\varsigma_{B,o}$ are calculated, the total mass transfer rates M can be

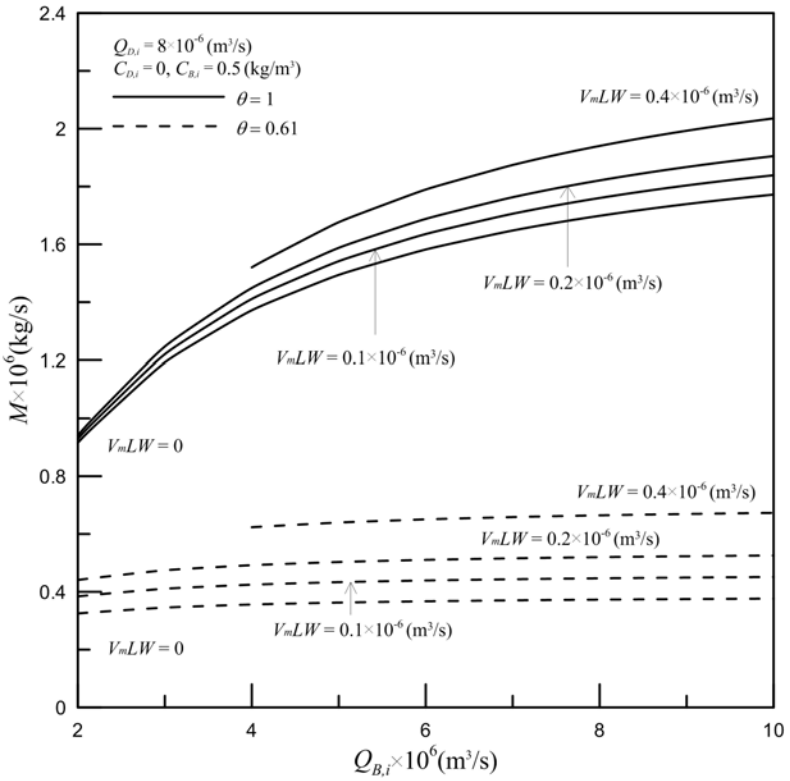

Fig. 3 Mass transfer rate for $Q_{D,i} = 8 \times 10^{-6}$ m³/s, $C_{B,i} = 0.5$ kg/m³, $C_{D,i} = 0$

obtained from Eq. (35). The results are given in Table 1 for $Q_{D,i} = 4 \times 10^{-6}$ m³/s and in Figs. 2 and 3 for $Q_{D,i} = 2 \times 10^{-6}$ m³/s and 8×10^{-6} m³/s, respectively. As expected the mass transfer rate increases when the inlet flow rates ($Q_{B,i}$ and $Q_{D,i}$) increase due to the dialysis effect, or when the ultrafiltration rate (V_mLW) increases due to the ultrafiltration effect. The increase in M with $Q_{B,i}$ or $Q_{D,i}$ is more sensitive for urea ($\theta = 1$) than for inulin ($\theta = 0.61$), while the increase in M with (V_mLW) is more noticeable for inulin than for urea because the mass transfer coefficient of the former in blood is lower than that of the later, giving more space for improved performance. The mass transfer rate of solute in urea solution is higher than in inulin solution because of the higher mass transfer coefficient, as well as the larger sieving coefficients.

3.2.3 Improvement in separation by ultrafiltration

The operation of ultrafiltration indeed enhances the mass transfer rate in dialysis operation, as shown in Table 1 for $Q_{D,i} = 4 \times 10^{-6}$ m³/s and in Figs. 4 and 5 for $Q_{D,i} = 2 \times 10^{-6}$ m³/s and 8×10^{-6} m³/s respectively. The improvement in mass transfer rate I increases when ultrafiltration rate increases or when the volumetric flow rates, $Q_{B,i}$ and $Q_{D,i}$, increase. For the dialysis system, which has lower dialysis rate, there has larger space for improving mass transfer rate by applying the ultrafiltration operation. Therefore, the comparison of I between urea ($\theta = 1$) and inulin ($\theta = 0.61$) solutions is: inulin in blood > urea in blood.

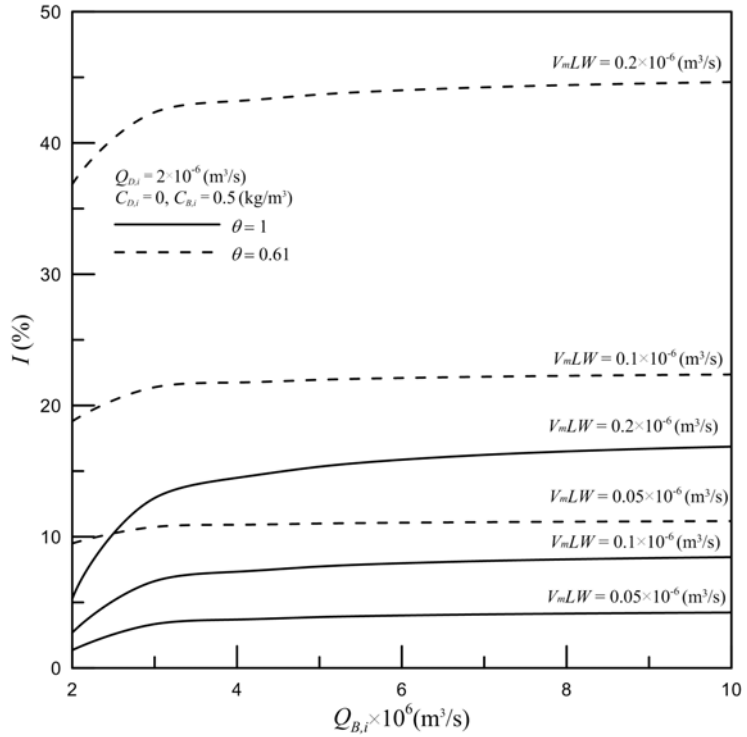

Fig. 4 Improvement in performance for $Q_{D,i} = 2 \times 10^{-6}$ m³/s, $C_{B,i} = 0.5$ kg/m³, $C_{D,i} = 0$

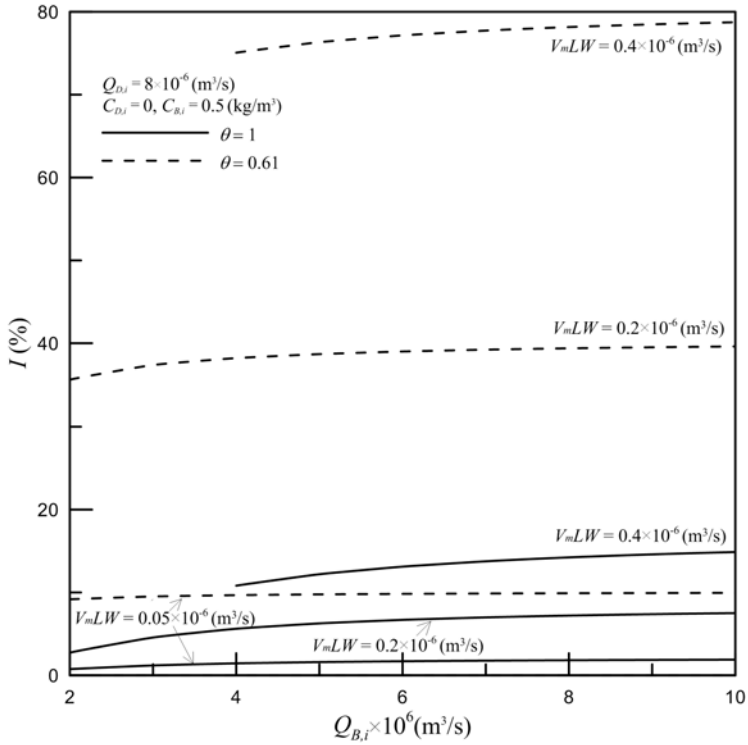

Fig. 5 Improvement in performance for $Q_{D,i} = 8 \times 10^{-6}$ m³/s, $C_{B,i} = 0.5$ kg/m³, $C_{D,i} = 0$

4. Conclusions

Applying the ultrafiltration operation to dialysis process in countercurrently parallel-flow rectangular membrane modules has been investigated. The solute concentration distributions in the retentate and dialysate phases were solved from mass balances with the assumption of uniform ultrafiltration flux for the operation with slight concentration polarization and declination of transmembrane pressure. Once the solute outlet concentrations were obtained, the mass transfer rates are readily calculated under various values of parameters, such as sieving coefficient θ, ultrafiltration rate (V_mLW) and volumetric flow rates ($Q_{B,i}$ and $Q_{D,i}$). The enhancement in separation efficiency of a dialysis process in countercurrently parallel-flow membrane module is substantially achievable if the operation of ultrafiltration is applied. It was found that the enhancement in mass transfer is significantly increased with increasing ultrafiltration flux, especially for the systems with low mass transfer coefficient. It may be observed in Table 1 and Figs. 2-5 that increasing the volumetric flow rate in retentate phase is more beneficial to mass transfer than increasing in dialysate phase. Moreover, it was reported (Yeh and Chang 2005) that the hydraulic-dissipated powers in a dialyzer are too small to be concerned about even under double-pass operation. Therefore, the increases in maintenance costs for ultrafiltration operation may be also ignored.

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Nomenclature

A	defined by Eq. (20)
a	numbers of transfer unit in retentate phase (LWK / $Q_{B,i}$)
B	defined by Eq. (24)
b	numbers of transfer unit in dialysate phase (LWK / $Q_{B,i}$)
C	solute concentration (kg/m^3)
D	defined by Eq. (25)
I	improvement in performance defined by Eq. (36)
K	overall mass transfer coefficient (m/s)
L	membrane length (m)
M	mass transfer rate (kg/s)
Q	volumetric flow rate (m^3/s)
Q_B	Q in retentate phase (m^3/s)
Q_D	Q in dialysate phase (m^3/s)
V_m	ultrafiltration flux (m/s)
W	membrane width (m)
x	coordinate (m)

Greek Symbols

m	defined by Eq. (29)
n	defined by Eq. (30)
ε	ratio of inlet solute concentration in dialysate phase to that in retentate phase ($C_{D,i}$ / $C_{B,i}$)
ς	dimensionless solute concentration (C / $C_{B,i}$)
θ	sieving coefficient
ξ	(x / L)
ϕ	(V_m / K)