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Parameter estimation of four-parameter viscoelastic Burger model by inverse analysis: case studies of four oil-refineries

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Abstract. This paper reports the development of a generalized inverse analysis formulation for the parameter estimation of four-parameter Burger model. The analysis is carried out by formulating the problem as a mathematical programming formulation in terms of identification of the design vector, the objective function and the design constraints. Thereafter, the formulated constrained nonlinear multivariable problem is solved with the aid of fmincon: an in-built constrained optimization solver module available in MatLab. In order to gain experience, a synthetic case-study is considered wherein key issues such as the determination and setting up of variable bounds, global optimality of the solution and minimum number of data-points required for prediction of parameters is addressed. The results reveal that the developed technique is quite efficient in predicting the model parameters. The best result is obtained when the design variables are subjected to a lower bound without any upper bound. Global optimality of the solution is achieved using the developed technique. A minimum of 4-5 randomly selected data-points are required to achieve the optimal solution. The above technique has also been adopted for real-time settlement of four oil refineries with encouraging results.

Keywords: inverse analysis; four-parameter Burger model; mathematical programming; constrained non-linear multivariable problem; variable bounds; global optimality.

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1. Introduction

Modeling and model parameter estimation is of paramount importance for a correct prediction of the foundation behavior. The primary aspect of foundation behavior rests with the understanding of the interaction between the foundation soil and the supported superstructure, which is still stated as the emerging topic in geotechnical engineering due to the lack of any universal consensus. Probably this notion is attributed to the complex behavior of soil both with space and with time. Several researchers around the world have provided evidence of the variety in the soil-structure interaction studies (Zolghadr Jahromi *et al*. 2007, Rajashekhar Swamy *et al*. 2011). In soil-structure interaction studies, lumped parameter modeling, made up of interconnected springs and dashpots, is a

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convenient way to represent the behavior of the foundation soils (Chore *et al*. 2010). Though four-parameter Burger model represents the soil behavior more realistically in comparison to other lower-order models, the use of this model for predictive purpose is not frequent, probably due to the difficulty in determining the parameters. Dey and Basudhar (2010) demonstrated the efficacy of four-parameter Burger model in predicting the response of viscoelastic foundation beds, and validated the same with the aid of a case study. However, the article reported a trial-and-error scheme to determine the model parameters (two elastic and two viscoelastic parameters). For known values of model parameters, a forward analysis can easily predict the response of the system. On the contrary, the determination of the model parameters from the inverse analysis of the observed system response is complex. Very often, such inverse analysis problems are ill-posed characterized by the presence of several solutions giving identical responses. Hence, the determined model parameters become the artifacts providing the response of the system as observed, although, in some cases they might not have physical significance. An iterative inverse-analysis technique combined with optimization algorithm may provide satisfactory solution to the above problem.

Conventionally, in a viscoelastic model, coefficients of viscous dashpots are functions of time-dependent parameters of soil, such as the degree of consolidation. Several researchers have made use of arbitrary dashpot coefficients in the formulation and analysis (Freudenthal and Lorsch 1957, Hoskin and Lee 1959, Murayama and Shibata 1961, Freudenthal and Spillers 1962, Schiffman *et al*. 1964, Bandopadhyay 1982); however, they did not suggest any generalized technique for the reasonable determination of the model parameters. Few simple techniques have been adopted by different researchers for simpler models. Taylor (1942) proposed Theory-B of consolidation and employed the ratio of incremental pressure to the induced settlement, to determine the coefficient of viscosity of a homogeneous viscoelastic medium using the data obtained from laboratory consolidation test. Lo (1961) performed long-term consolidation tests on remolded and undisturbed clay samples to determine the time-settlement behavior of cohesive soils due to secondary consolidation and proposed two different methodologies to determine the rheological parameters for a three-parameter model. The first method uses a ratio of incremental settlement with respect to incremental stress. The second method uses a plot of logarithmic incremental settlement to time, subjected to a constant incremental time and a constant viscoelasticity ratio. Schultze and Krause (1964) carried out large-scale compression tests to determine the magnitude of the residual pore water pressure, and developed isochronous curves for various time durations to provide qualitative rheological prediction about the viscosity of clayey soil. Juskiewicz-Bednarczyk and Werno (1981) presented the determination of one-dimensional consolidation parameters for a viscoelastic medium represented by a three-element lumped parameter mechanical model. The coefficient of viscosity and other consolidation parameters were determined with the aid of an iterative procedure applying minimization of an error function with respect to the unknown parameters. It is recognizable that the applicability of the above techniques becomes difficult with the increase in the complexity of the model. At the same time, complexity of a model, to a limited extent, is required, for the proper representation of the constitutive behavior of the medium; for e.g. at least four-element lumped parameter model is required for the proper representation of a viscoelastic medium (Dey and Basudhar 2010). Under such conditions, adoption of inverse analysis, aided with modern computing techniques applying optimization and neural networks, to determine the parameters from the stress-deformation-time response of a viscoelastic medium from laboratory or in-situ investigations and likely to be more computationally efficient.

This paper reports the development of a generalized methodology to determine the model parameters of a four-element Burger model from the observational behavior of a viscoelastic soil medium. In this regard, this article provides a brief description and the background of inverse analysis, followed by the development of the mathematical programming formulation of the problem. In order to gain experience and judge the efficacy of the developed technique, based on a synthetic case study, the article furnishes a discussion regarding the prime issues related to such problem. Thereafter, the article provides a report on the application of the developed methodology to the time-dependent settlements of oil tanks in four different oil refineries.

2. Parameter estimation by inverse analysis

Inverse analysis is a convenient approach to determine the model parameters using observational data gathered from either laboratory experiments (Li *et al*. 2000) or in-situ investigations (Gioda and Locatelli 1999).

Research pertaining to the inverse analysis of parameters in the field of geophysics and ground water flow are available in plenty. From a quite large number of literature, it is evident that a considerable number of applications have also been made in the field of geotechnical engineering, in the vistas of consolidation (Kim and Lee 1997), rock mechanics (Gioda 1985), slope stability (Bhattacharya and Basudhar 1996), retaining structures (Gioda and Jurina 1981), load-settlement behavior and bearing capacity of subgrade (Bergado *et al.* 1992), pile-settlement analysis (Honjo *et al*. 1993) and tunneling works (Gioda and Locatelli 1999). Although well established in geotechnical engineering, an ad hoc use of inverse-analysis to determine the soil and rock parameters raises several controversial issues (Leroueil and Tavenas 1981). In the modern era, the application of optimization techniques and neural networks has opened the gateways for more and rational approaches (Raju 1999).

The solution techniques for the inverse analysis may be based on closed form solutions (Kirsten 1976), numerical iterative procedures (Shoji *et al*. 1990), probabilistic and statistical approaches (Li *et al*. 2000) or adoption of special optimization techniques (Ledesma *et al*. 1989).

The above-mentioned methods estimate the parameters of the model by determining those values that provide an optimal match of the output of the model and the output of the real system, for the same input. A scalar measure-of-fit determines the quality of the match. The purpose of the parameter adjustment algorithm is to select the optimal parameter values by minimizing the measure-of-fit in a systematic procedure. Thus, the optimization algorithm plays a key role in the inverse analysis. Fig. 1 depicts the sequential steps of an inverse analysis technique in conjunction with the optimization algorithm.

Thus, herein, the development of a generalized inverse analysis technique is presented, formulating the problem as one that of mathematical programming; with the appropriate identification of the design vector, the objective function and the design constraints. The present study reports the use of fmincon: an in-built constrained optimization solver module available in MatLab, to solve the formulated constrained problem.

3. Problem definition

Fig. 2 depicts the rheological sketch of the standard four-element viscoelastic Burger model

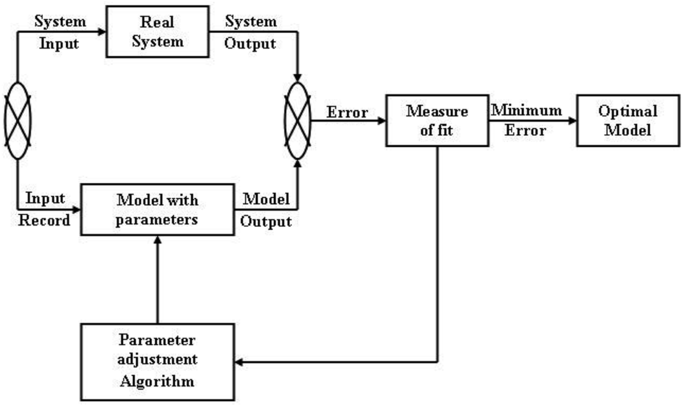

Fig. 1 Algorithm to determine the model parameters by inverse analysis

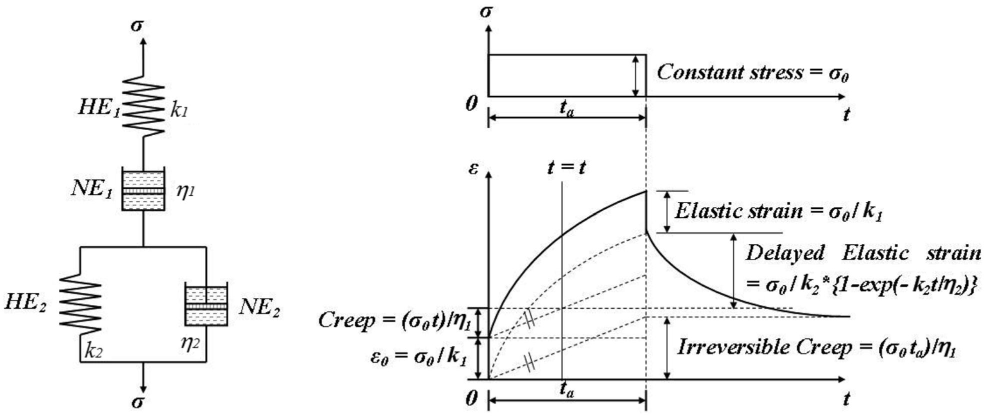

Fig. 2 Rheological model and the response of a four-parameter Burger element

subjected to a constant stress (σ) for a particular time-period (t_a) and the corresponding stress-strain behavior with time showing the different phases of response. The problem is to estimate the model parameters (k_1, k_2, η_1, η_2) from the measured deformation-time behavior of a viscoelastic medium. The suitability and efficacy of four-parameter Burger model in representing the time-dependent phenomena of viscoelastic soil has been furnished by Dey and Basudhar (2010).

4. Analysis

The following sections describe the formulation in terms of the identification of the design vector, the objective function and the design constraints, and the subsequent development of the mathematical programming formulation. The four-parameter Burger model is capable of representing both loading and unloading behavior of a viscoelastic medium. However, the efficacy of the developed procedure in back-estimating the model parameters has been demonstrated considering only the loading phase.

4.1 Design variables [$\{k_1, k_2, \eta_1, \eta_2\}$] and design vector [$\vec{V}$]

For the present study, the design variables governing the constitutive behavior of the Burger model are as follows: (1) Elastic coefficient of the Maxwell sub-unit (k_1), (2) Elastic coefficient of the Kelvin sub-unit (k_2), (3) Viscous coefficient of the Maxwell sub-unit (η_1) and (4) Viscous coefficient of the Kelvin sub-unit (η_2). Thus, the design vector becomes

$$\vec{V} = \{k_1, k_2, \eta_1, \eta_2\}^T \tag{1}$$

4.2 Objective function [$F(\vec{V})$]

The choice of the objective function should be such that it reflects the difference between the measured and the predicted values. Thus, considering the measured and predicted values of the deformation at any instant of time, using the method of least squares, the normalized form of the objective function is

$$F(\vec{V}) = \sum_{n_p = 1}^{n_p} \left(\frac{\varepsilon_{expr}(t_{n_p}) - \varepsilon_{BU}(t_{n_p})}{\varepsilon_{expr}(t_{n_p})} \right)^2 \tag{2}$$

where, n_p is the number of data-points considered for inverse analysis, t_{n_p} is the elapsed time corresponding to the considered data-point, $\varepsilon_{expr}(t_{n_p})$ is the measured magnitude of deformation at the instant of time corresponding to the considered data-point, and $\varepsilon_{BU}(t_{n_p})$ is the predicted magnitude of deformation at the instant of time corresponding to the considered data-point and is defined as

$$\varepsilon_{BU}(t_{n_p}) = \sigma_0 \left[\frac{1}{k_1} + \frac{t_{n_p}}{\eta_1} + \frac{1}{k_2} \left\{ 1 - \exp\left(-\frac{k_2 t_{n_p}}{\eta_2} \right) \right\} \right] \tag{3}$$

Thus, the objective function, as expressed by Eq. 2, is nonlinear and multivariable in nature.

4.3 Design constraints

In order to ensure that the estimated magnitudes of the design variables are physically reasonable and acceptable and to reduce the search domain, the variable bounds imposed as constraints on the model parameters are

$$k_1^{\min} \le k_1 \le k_1^{\max},\ k_2^{\min} \le k_2 \le k_2^{\max},\ \eta_1^{\min} \le \eta_1 \le \eta_1^{\max},\ \eta_2^{\min} \le \eta_2 \le \eta_2^{\max} \tag{4}$$

where, $v_i^{\min}, v_i^{\max}$ designates the minimum and the maximum bounds on the design parameter v_i. The standard form of expressing the constraints is

$$(v_i^{\min} - v_i) \le 0.0,\ (v_i - v_i^{\max}) \le 0.0 \quad \forall i = 1, 2, 3, 4 \tag{5}$$

Hence, the imposed constraints are all linear.

4.4 Mathematical programming formulation

The generalized standard format of representing the constraints (Eq. 4) for the mathematical programming formulation is

$$g_i(\vec{V}) \leq 0.0, \quad i = 1, 2, 3, 4 \tag{6}$$

Minimization of the objective function with respect to the design variables results in the determination of the optimal value. Hence, the optimization problem for the present study is

$$\left.\begin{array}{c} \text{Find optimal design variable vector } (\vec{V}_{optim}), \textit{such that}, \\ F = F(\vec{V}_{optim}) \textit{ is the minimum of } F(\vec{V}) \\ \text{subject to} \\ g_i(\vec{V}) \leq 0.0, \quad i = 1, 2, \ldots, 8 \end{array}\right\} \tag{7}$$

In the above formulation, the objective function and the design constraints, respectively being nonlinear and linear functions of the design variables, the problem is one of nonlinear programming.

5. Solution technique: solving a nonlinear constrained optimization problem

The present study uses fmincon to solve the constrained nonlinear multivariable optimization problem and estimate the parameters of the Burger model from the measured time-dependent behavior of a viscoelastic medium. The optimization scheme adopted in MatLab incorporates Sequential Quadratic Programming (SQP). At all the major iteration, an approximation is made of the Hessian of the Lagrangian function using a quasi-Newton method. This generates a Quadratic Programming (QP) sub-problem whose solution forms a search direction using a line search technique in order to proceed with the iterative process. Optimization Toolbox Users Manual (2008) provides a detailed description of the above method.

6. Results and discussions

6.1 Surface, contour plan and the nature of the objective function

Success of the developed technique depends largely on the proper understanding of the nature of the developed objective function. In this regard, the article reports a sensitivity study in order to examine the variation in the objective function with the basic model parameters (k_1, k_2, η_1, η_2). The other parameters governing the objective function are the applied stress (σ) and the elapsed time (t). However, these parameters are not a part of the set of design variables governing the objective function. Hence, the analysis considers them constants while determining the nature of the objective function.

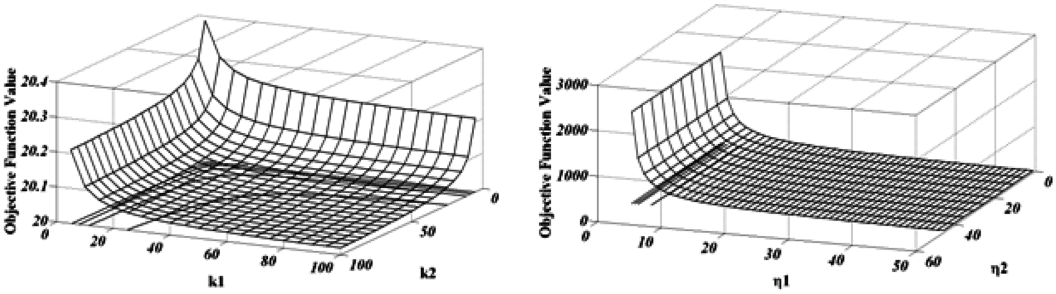

Fig. 3 Surface and contour plan of the objective function with respect to various combinations of contributory parameters

Since there are four design variables governing the objective function, it is not possible to depict the effect of all the parameters on the nature of the Burger function simultaneously in a single plot. Hence, for the sensitivity study, a series of plots are necessary considering two typical design parameters at a time while keeping the other two constant. Fig. 3 depicts two typical variations of the objective function with the corresponding design variables; for the sake of brevity, the article does not present the variation for the rest of the parametric combinations. Fig. 3 shows that for different combinations of the design parameters, the value of the objective function vary widely in the objective function space; this indicates significant influence of each of the parameter on the objective function. The contours of the objective function, obtained by varying any of the two design variables out of four, are very steep initially and thereafter very flat over a wide region.

These figures do not provide any precise idea about the nature of the objective function (due to the inability to plot the simultaneous effect of the four design parameters on the objective function). However, it is understandable that the simultaneous variation of all the four design parameters results in a complex hyper-surface in the design space that is difficult to visualize and present graphically. Thus, the difficulty in estimating the parameters increases manifold. This is mainly because the function is very flat over a wide region and several combinations of the design variables may result in nearly identical objective function value on reaching the plateau satisfying the convergence criterion.

6.2 A synthetic case study

The relevant model parameters, stress level and the time-period for the synthetic problem as chosen to gain experience for successful implementation of the developed methodology are

$k_1 = 30$ N/cm^2, $k_2 = 3$ N/cm^2, $\eta_1 = 3e4$ N-days/cm^2, $\eta_2 = 3e3$ N-days/cm^2, $t_a = 5000$ days and $\sigma = 20$ N/cm^2

Incorporating the above parameters, a forward analysis determines the stress-strain-time behavior of the Burger model. For a successful inverse analysis, it is essential to address the following issues:

(1) Determination and setting up of the variable bounds
(2) Global optimality of the obtained solution
(3) Minimum number of data-points required for correct parameter-estimation

3.2.1 Determination and the setting up of variable bounds

In an inverse analysis, the prediction of reliable values of parameters is highly dependent on two very important aspects namely:

(1) Choice of the feasible initial starting point and
(2) Bounds on the design variables

Knowledge of the approximate values of the design vector yields better prediction with less computational effort. As such, to get better approximation of the parameters, it may be necessary to furnish a-priori information gathered from data collected either by laboratory or in-situ tests. This information aids in the probable imposition of the approximate bounds on the parameters. Bhattacharya and Basudhar (1996) successfully adopted such a technique for the inverse-analysis of slopes. However, this methodology though being quite suitable while dealing with actual soil parameters, such a-priori information is not often adequately available for the model parameters associated with the lumped parameter models used in geotechnical engineering.

Nevertheless, the present study adopts the above technique in order to gain experience in determining the influence of bounds on the design variables. The experience, so gained, would subsequently help in choosing a suitable bound for the design variables in an unknown real-time set-up. Arbitrary representative sets of data (consisting of 2, 3, 4 and 5 number of data-points is chosen sequentially) are considered from the curve displayed in Fig. 4, and the inverse analysis is carried out to estimate the model parameters. For identical sets of initial design vectors, the analysis considers 10%, 20%, 50%, 100%, 200% and 400% slack on the chosen values of the design variables. These slacks correspond to the upper and lower bounds on the design variables. For the sake of brevity, the article enumerates only the 10% and 100% slacks in order to demonstrate the range of the design variables. Range of the parameters for the others slacks can be easily calculated.

(1) 10% bound on the actual design variables
$27 < k_1 < 33$, $2.7 < k_2 < 3.3$, $2.7e4 < \eta_1 < 3.3e4$, $2.7e3 < \eta_2 < 3.3e3$

(2) 100% bound on the actual design variables
$0 < k_1 < 60$, $0 < k_2 < 6$, $0 < \eta_1 < 6e4$, $0 < \eta_2 < 6e3$

For 100%, 200% and 400% bounds on the design variables, the lower bounds are chosen to be 0 (zero) as the design variables must be always positive. Such a wide variation may be necessary when no a-priori information about the model parameters is available. Table 1 reveals the effect of variable bounds on the magnitudes of the parameters.

Table 1 show that the variable bound chosen during the computation significantly affects the magnitudes of the estimated parameters. For a strict bound on the design variable vector, the solution obtained from the inverse analysis remains in vicinity to the actual design variables (as observed from the results for 10% bound). However, as the bound on the variable is relaxed, i.e., a wider bound is chosen, it is observed that the solution obtained from the inverse analysis differ largely from the actual solution (as observed for 50%, 100%, 200% and 400% slacks). This may be due to the restriction imposed by the constraints on the design space. In such a case, the assumed initial design vector may be feasible or infeasible according to its position within or outside the defined boundary respectively. Under such circumstances, at any step during the execution of the optimization scheme, one or the other constraint remains active, and significantly influence the determination of the optimal point. Moreover, the objective function forms a plateau over an

Table 1 Influence of variable bounds on the estimated parameters for the synthetic problem (Number of arbitrarily chosen data-points = 2)

Parameter	Initial design vector	Estimated values of parameters for different variable bounds						Actual design vector
		10%	20%	50%	100%	200%	400%	
k_1 (N/cm^2)	100	30.73	32.78	41.85	57.5	89.5	138	30
k_2 (N/cm^2)	100	3.19	3.45	4.5	6	9	15	3
η_1 (N - days/cm^2)	100	2.7e4	2.4e4	1.8e4	1.5e4	1.3e4	1.2e4	3e4
η_2 (N - days/cm^2)	100	2.7e3	2.4e3	1.5e3	100	100	100	3e3

extended region of the objective function space, and hence, various design variable vectors may result in nearly identical objective function value. The above condition may result in the observed deviation between the actual and estimated values of the model parameters. Such an effect is also due to the convergence criterion on the obtained results.

Thus, the above observation reveals that adoption of appropriate variable bounds is only possible when there is sufficient confidence on the a-priori information about the design variables. However, such information is generally not available for model parameters. Hence, in order to gain further insight, the present study reports an analysis without imposing strict bounds on the variables. As such, the analysis disregards any upper bounds on the design variables; however, it considers the variation in the lower bound on the convergence of the solution. The present study reports four different sets of lower bounds, which are:

(1) Set 1: $k_1 \geq 10$, $k_2 \geq 1$, $\eta_1 \geq 10000$, $\eta_2 \geq 1000$
(2) Set 2: $k_1 \geq 10$, $k_2 \geq 1$, $\eta_1 \geq 100$, $\eta_2 \geq 100$
(3) Set 3: $k_1 \geq 1$, $k_2 \geq 1$, $\eta_1 \geq 1$, $\eta_2 \geq 1$
(4) Set 4: $k_1 \geq 0$, $k_2 \geq 0$, $\eta_1 \geq 0$, $\eta_2 \geq 0$

Table 2 reveals that a nominal positive value as lower bound for the design variables is sufficient for obtaining meaningful results. A zero value of lower bound generates complex numbers and results in the premature termination of the optimization scheme. Such an event occurred as the mathematical expression of the objective function contains design variables as denominators. During the optimization process, due to the chosen lower bound, if any of such design variables is assigned to be 0 (zero), the value of the objective function becomes infinite. Hence, in order to avoid such criticality, all the subsequent studies consider $[1\ 1\ 1\ 1]^T$ as the lower bound of the design variables. However, this bound may change depending on the specific problem, and as such, any optimization scheme, before execution, must determine the same for that particular problem.

6.2.2 Global optimality of the obtained solution

The present study considers four initial design vectors in order to establish the global optimality of the solutions obtained from the inverse analysis. The different sets of initial design vectors are:

(1) Set 1: $k_1 = 1$, $k_2 = 1$, $\eta_1 = 1$, $\eta_2 = 1$, i.e., identical to the lower bound vector
(2) Set 2: $k_1 = 10$, $k_2 = 10$, $\eta_1 = 10$, $\eta_2 = 10$
(3) Set 3: $k_1 = 100$, $k_2 = 100$, $\eta_1 = 100$, $\eta_2 = 100$
(4) Set 4: $k_1 = 100$, $k_2 = 100$, $\eta_1 = 10000$, $\eta_2 = 1000$

Table 2 Influence of different lower bounds on the estimated parameters for the synthetic problem (Number of arbitrarily chosen data-points = 5)

Parameter	Initial design vector	Estimated values of parameters for different lower bound on design variables				Actual design vector
		Set 1	Set 2	Set 3	Set 4	
k_1 (N/cm^2)	100	30	30	30	Complex magnitudes	30
k_2 (N/cm^2)	100	3	3	3		3
η_1 (N - days/cm^2)	100	3e4	3e4	3e4		3e4
η_2 (N - days/cm^2)	100	3e3	3e3	3e3		3e3
Objective function value	-	2.5e-13	2.5e-13	2.5e-13	Premature termination	

Table 3 Influence of initial design vector on the estimated parameters for the synthetic problem (Number of arbitrarily chosen data-points = 5)

Parameter	Estimated values of parameters for different initial design variable vectors				Actual design vector
	Set 1	Set 2	Set 3	Set 4	
k_1 (N/cm^2)	30.03	29.5	30	30	30
k_2 (N/cm^2)	3.1	3	3.3	2.94	3
η_1 (N - days/cm^2)	3e4	3e4	3e4	3e4	3e4
η_2 (N - days/cm^2)	2.9e3	3e3	3e3	3e3	3e3
Objective function value	2.6e-13	2.5e-13	2.5e-13	2.32e-13	
Number of function evaluations	1368	513	259	168	

Table 3 reveals the results from the inverse analyses, which shows that all the four design vectors led to nearly identical objective function values, and the estimated parameters are identical to the values of the model parameters as well. Thus, it shows that a globally optimum solution is available for the problem. However, Table 3 reveals that the choice of initial vector affects the computational time required to reach the optimum.

6.2.3 Minimum number of data-points required for parameter-estimation

In general, for determining the model parameters, it is necessary and sufficient to have number of observational points equal to the number of unknown parameters. Thus, in estimating the parameters, very often, one may encounter situations when the number of observational data points may be more, equal or less than the number of the parameters to be determined making the system to be either overdetermined or underdetermined, if not exact. The number of model parameters for the present study is four. Therefore, if the data points are precisely determined, 4 points should give correct solution. However, it is agreeable that such precise selection of observational data-points is not generally possible from load-deformation-time behavior obtained from real-time investigations. Hence, for the present study, it is necessary to find the minimum number of data points that would provide reasonably good results with less computational effort. Table 4 enumerates the results from the analyses considering 2 - 5 data points, which forms the observational data-sets as follows:

(1) Set 1: 2 observational data-points: t_{n_p} = 0, 5000 days

(2) Set 2: 3 observational data-points: t_{n_p} = 0, 2500, 5000 days

(3) Set 3: 4 observational data-points: t_{n_p} = 0, 1250, 3750, 5000 days

(4) Set 4: 5 observational data-points: t_{n_p} = 0, 1250, 2500, 3750, 5000 days

(5) Set 5: Randomly selected 4 observational data-points: t_{n_p} = 1000, 2250, 3600, 4000 days

This section reports the results of the parameter estimation considering sequential selection of the above observational data-sets. For the first four cases, the deformation corresponding to the time of commencement and removal of loading is included along with intermediate points, wherein the overall sets consists of data-points at regular intervals. The primary aim of these choices was to capture the overall constitutive behavior of the model. However, for the last case, in order to check the versatility and capability of inverse analysis technique in determining of the model parameters, the dataset comprises of four observational data-points selected at random from the strain-time representation.

Table 4 reveals that the prediction of the elastic constant (k_1) is quite accurate for all the four cases, when the strain at the instant of loading ($t_{n_p} = 0$) is included as one of the observational data-point. However, while analyzing a problem it may not be always possible to have the settlement data at the instant of loading. Moreover, even when done very precisely, an accurate measurement of settlement at the instant of loading depends on various factors such as the instrumentation errors, non-verticality of the load, and contact surface imperfection between the load and the subgrade. As such, the observational data-point vector for the fifth set comprises of randomly chosen points, excluding the data-point at $t_{n_p} = 0$, to check the capability of the adopted inverse analysis technique to provide a proper estimation. Table 4 shows that, even under such circumstances, estimation of parameters is excellent, as indicated by the negligible value of the Root Mean Square Error (RMSE - a commonly used error criteria to judge the nearness of the outcome of a problem to its exact solution) difference between the chosen and estimated results. However, it is noted that under such circumstances, the elastic parameter (k_1) is not identical to the chosen parameter; however, the difference is negligibly small (< 0.05%). As such, for all practical purpose the results are the same.

Table 4 Influence of number of observational data-points on the estimated results of the synthetic problem

Parameter	Estimated values of parameters for different number of observational data-points					Actual design vector
	Set 1	Set 2	Set 3	Set 4	Set 5	
k_1 (N/cm^2)	30	30	30	30	29.99	30
k_2 (N/cm^2)	317.98	3.56	3	3	3	3
η_1 (N - days/cm^2)	1.01*e*4	2.3*e*4	2.9*e*4	3*e*4	3*e*4	3*e*4
η_2 (N - days/cm^2)	1*e*3	1*e*3	3*e*3	3*e*3	3*e*3	3*e*3
Objective function value	2.5*e*-13	2.5*e*-13	2.5*e*-13	2.32*e*-13	2.32*e*-13	
RMSE of the chosen and estimated values	531.9	995.3	4e-5	0	5e-6	
Number of function evaluations	164	9415	841	494	500	

Table 4 reveals that with the increase in the number of observational data-points, the RMSE error decreases; and, the error vanishes for five observational data-point set. However, with four points, considering data at either regular or randomly chosen time intervals, the errors are also negligibly small. Thus, this study reveals that the minimum number of observational data-points required for a proper estimation of the model parameters is equal to 4, which is theoretically correct.

With the experience thus gained, the following section reports the efficacy of the developed method extended to the real-time cases of laboratory and field investigations.

6.3 Case-studies

Several case studies from the literatures involving the settlement of a viscoelastic clay medium under a constant stress applied for a specific period have been analysed. Using the data from the literatures, the inverse analysis technique has been used to determine the parameters of the Burger model. Once the bask-estimation has been done, a forward analysis has been carried out to determine the degree of correspondence between actual and predicted results. Out of the several studies, the present report describes the application of inverse analysis to the time-dependent deformation of the oil tanks in four different refineries.

6.3.1 Settlement during the preloading for the construction of a fuel-oil tank at Avon refinery, San Francisco, California monitored for a period of 4 months (Darragh (1964))

Darragh (1964) reported the time-settlement results obtained during the preloading of the subsoil for the construction of fuel-oil tank at Avon refinery, San Francisco, California. The subsoil consisted of peaty clays for a depth of 1 m-1.33 m, and underlain by 1.33 m-3 m of moderately compressible clays intermixed with sands. The bed-soil consisted of 16 m of stiff clays. The soft soils had a moisture content of 200%-400%. Ground water level was found within few inches of the ground surface. The preloading was achieved using the technique water preloading of the tanks, wherein the preloading was applied by a water height of 9 m for a period of 4 months. Inverse analysis carried out with the relevant data revealed that a minimum number of 4 data-points are necessary for the determination of proper model parameters, and they are obtained as follows

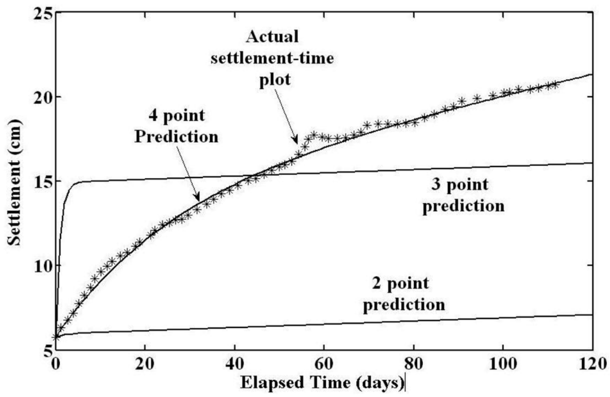

Fig. 4 Comparison of the actual and predicted settlement-time behavior from the results of preloading for the construction of a fuel-oil tank at Avon refinery, San Francisco, California monitored for a period of 4 months (Actual behavior obtained from Darragh (1964))

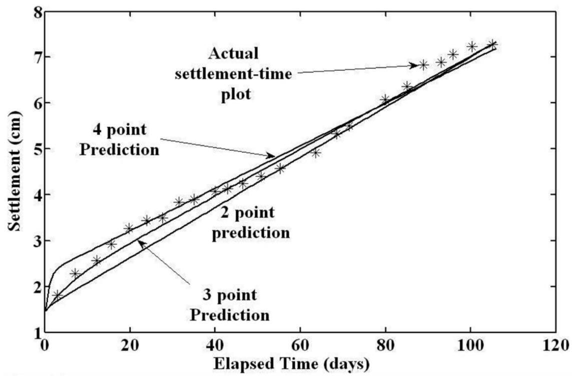

Fig. 5 Comparison of the actual and predicted settlement-time behavior from the results of preloading for the construction of a mobil-oil tank at Oakland, California monitored for a period of 4 months (Actual behavior obtained from Darragh (1964))

$k_1 = 1.58$ N/cm^2, $k_2 = 1.079$ N/cm^2, $\eta_1 = 1.52e2$ N-days/cm^2, $\eta_2 = 28.1$ N-days/cm^2

Objective function value = 1.69e-17 and Number of function evaluations = 7415.

Fig. 4 depicts the result of the forward analysis carried out using the parameters obtained from the inverse analysis considering different numbers of observational data-points.

6.3.2 Settlement during the preloading for the construction of a mobil-oil tank at Oakland terminal, California monitored for a period of 4 months (Darragh (1964))

Darragh (1964) reported the time-settlement results obtained during the preloading of the subsoil for the construction of mobil-oil tank at Oakland terminal, California. The subsoil consisted of 2.7 m of fill and 3 m-4.3 m of soft silty and sandy clays underlain by 15 m of dense sand followed by stiff to hard over-consolidated clay. The soft clays consisted of post-glacial marine deposits, locally referred as bay-mud. The preloading was achieved using the technique water preloading of the tanks, wherein the preloading was applied by a water height of 11 m for a period of 4 months. Inverse analysis carried out with a set of data for an intermediate period revealed that a minimum number of 4 data-points are necessary for the determination of proper model parameters, and they are obtained as follows

$k_1 = 0.39$ N/cm^2, $k_2 = 9.4$ N/cm^2, $\eta_1 = 173.7$ N-days/cm^2, $\eta_2 = 7.8$ N-days/cm^2

Objective function value = 2.1e-3 and Number of function evaluations = 564.

Fig. 5 depicts the result of the forward analysis carried out using the parameters obtained from the inverse analysis considering different numbers of observational data-points.

6.3.3 Settlement during the preloading for the construction of storage tanks at Refineria Panama, South America monitored for a period of 35 days (Darragh (1964))

Darragh (1964) reported the time-settlement results obtained during the preloading of the subsoil for the construction of storage tanks at Refineria Panama, South America. The subsoil consisted of 1.2 m thick coral sand fill underlain by 1.2 m thick peat bed followed by 6 m-18 m of coral formation. The preloading was achieved using the technique water preloading of the tanks, wherein the preloading was applied by a water height of 13 m for a period of 35 days. Inverse analysis

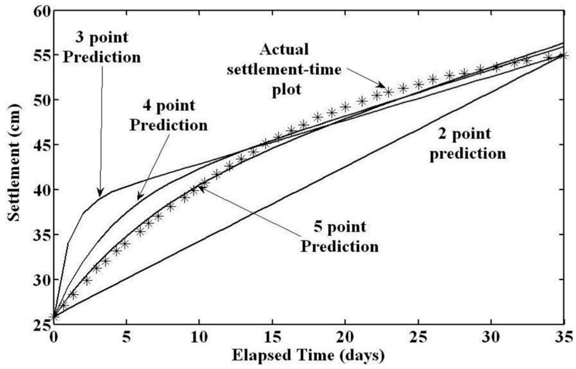

Fig. 6 Comparison of the actual and predicted settlement-time behavior from the results of preloading for the storage tanks at Refineria Panama, South America monitored for a period of 35 days (Actual behavior obtained from Darragh (1964))

carried out with the relevant data revealed that a minimum number of 5 data-points are necessary for the determination of proper model parameters, and they are obtained as follows

$k_1 = 0.472$ N/cm^2, $k_2 = 23$ N/cm^2, $\eta_1 = 6.7$ N-days/cm^2, $\eta_2 = 0.472$ N-days/cm^2

Objective function value = 1.8e-3 and Number of function evaluations = 1344.

Fig. 6 depicts the result of the forward analysis carried out using the parameters obtained from the inverse analysis considering different numbers of observational data-points.

6.3.4 Settlement during the preloading for the construction of storage tanks at Pascagoula Refinery, Kentucky monitored for a period of 9 months (Darragh (1964))

Darragh (1964) reported the time-settlement results obtained during the preloading of the subsoil for the construction of storage tanks at Pascgoula Refinery, Kentucky. The subsoil consisted of uniformly compacted silty sands of thickness 10 m, underlain by 5 m of soft compressible clays and underlain by 6 m of medium stiff sandy clay and compacted clayey sands, and 6 m of stiff clays with moderate compresisblity. The preloading was achieved using the technique water preloading of the tanks, wherein the preloading was applied by a water height of 13 m for a period of 9 months. Inverse analysis carried out with the relevant data revealed that a minimum number of 3 data-points are necessary for the determination of proper model parameters, and they are obtained as follows

$k_1 = 2.859$ N/cm^2, $k_2 = 1$ N/cm^2, $\eta_1 = 158$ N-days/cm^2, $\eta_2 = 105$ N-days/cm^2

Objective function value = 1.61e-16 and Number of function evaluations = 9420.

Fig. 7 depicts the result of the forward analysis carried out using the parameters obtained from the inverse analysis considering different numbers of observational data-points. A very good agreement is observed between the actual and the predicted results obtained from a 3-point inverse analysis.

7. Probable range of Burger model parameters as obtained from inverse analysis of case studies

Based on several case studies, a wide variety of viscoelastic soils of varying properties have been

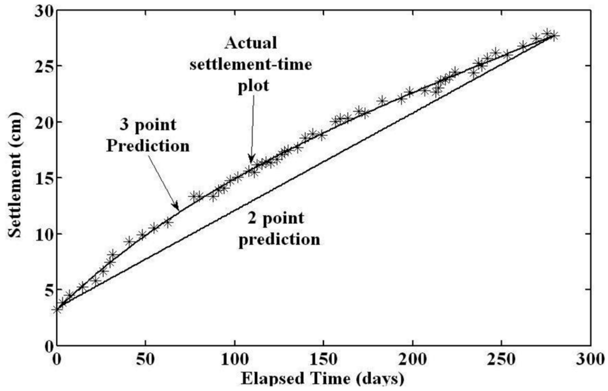

Fig. 7 Comparison of the actual and predicted settlement-time behavior from the results of preloading for the storage tanks at Pascagoula Refinery, Kentucky monitored for a period of 9 months (Actual behavior obtained from Darragh (1964))

studied to determine the model parameters by inverse analysis. Moreover, settlement-time data of the different soils have been obtained from widely differing monitoring periods, ranging from 50 hours to 38 years. Thus, it can be suitably inferred that the results, obtained considering such wide variations, would be quite capable in describing the range of model parameters of the Burger element when used for the representation of viscoelastic medium. The ranges of the different model parameters obtained from inverse analysis of the case studies are enumerated as follows

$$k_1 : 0.3 - 10^4 \text{ N/cm}^2,\ k_2 : 1 - 3520 \text{ N/cm}^2$$

$$\eta_1 : 6.7 - 2 \times 10^8 \text{ N-days/cm}^2,\ \eta_2 : 0.472 - 10^4 \text{ N-days/cm}^2$$

These parameters govern the constitutive behavior of the Burger model, and hence various combinations of these parameters can be chosen in representing the settlement-time behavior of a viscoelastic subgrade subjected to constant stress for a particular period. The relative magnitudes of the various parameters can be enumerated to be in the range as follows

$$\frac{k_1}{k_2}: 0.0045 - 9.91,\ \frac{\eta_1}{\eta_2}: 1.5 - 3.9 \times 10^5,\ \frac{k_1 \eta_1}{k_2 \eta_2}: 0.032 - 3.8 \times 10^6$$

Depending on the ratios of the various model parameters as described above, the constitutive behavior of the Burger model and all of its degenerative models can be represented. A low magnitude of the ratio, k_1/k_2, degenerate the behavior of the Burger to represent the constitutive behavior of the Maxwell model, whereas, a high value represents the behavior of a three-parameter fluid model. Similarly, a low value of η_1/η_2 reflects the behavior of Maxwell element, whereas a high value of the ratio of viscous constants degenerate the Burger model to a Poynting-Thompson element. A high magnitude of $\frac{k_1 \eta_1}{k_2 \eta_2}$ reflects the behavior of a Maxwell element, whereas a low magnitude of the composite ratio reveals the behavior of the Kelvin-Voigt element. The behavior of the Burger model is appreciated for intermediary values of the above-mentioned ratios.

8. Limitation of the proposed approach

The proposed approach of using inverse analysis for estimating the parameters of Burger model is primarily intended for understanding and deducting the behavior of viscoelastic soils. It has been mentioned in the present article that Burger model is largely capable of representing the behavior of a viscoelastic soil when considered in terms of the instantaneous settlement under loading followed by a creep phase under sustained loading. When unloaded, it can simulate the behavior of instantaneous recovery followed by sustained relaxation. However, the authors would like to mention here that all the investigations reported herein are solely based on the loading behavior of the viscoelastic soils. Since all the data acquired are primarily real-time with most of the project are and have been functional, it had not been possible for the authors to collect the unloading or reloading data for the same soils. The collected data definitely restricts the present approach of estimating the Burger model parameters being valid only for the loaded viscoelastic soils. The primary intention of the project has been to investigate the Burger model parameters in terms of the loading-unloading-reloading behavior of the viscoelastic soils. Such behavior, as the authors believe, can be obtained from compression and swelling behaviors of viscoelastic soils which can be performed in an oedometer in the laboratory. In the field, this can be carried out by a controlled preloading scheme wherein the viscoelastic soil is preloaded and later the preloading is dismantled for actual construction. Hence, the present study does not ensure that the Burger model parameters estimated using inverse analysis would also be valid for the unloading and reloading sequences. Hence, the proposed approach limits its usage only under loading sequence of viscoelastic soils. There is need for further research to investigate the possible and probable change in the model parameters during the unloading and reloading sequences.

9. Conclusions

Based on the above studies, the following conclusions are drawn:

- The developed scheme for model parameter estimation using Inverse analysis technique in conjunction with an efficient optimization scheme has been found to be very effective.
- The bounds on the model parameters significantly affect the estimated parameters. Best results are obtained when the upper bound on the design variables is eliminated with a nominal lower bound on the same.
- Use of the scheme ensured global optimality of the solution. However, strict convergence criterion should be imposed to remove the possibility of premature termination.
- Minimum of 4-5 numbers of randomly selected observational data-points are required for achieving the optimal solution.
- The efficacy of the developed methodology was scrutinized by considering several case studies. For all the cases, the developed technique yielded model parameter values, which when used in a forward analysis scheme, predicts the time-settlement behavior of the viscoelastic medium, exhibiting excellent match with the observed behavior.

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