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The effects of stiffness strengthening nonlocal stress and axial tension on free vibration of cantilever nanobeams

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Abstract. This paper presents a new nonlocal stress variational principle approach for the transverse free vibration of an Euler-Bernoulli cantilever nanobeam with an initial axial tension at its free end. The effects of a nanoscale at molecular level unavailable in classical mechanics are investigated and discussed. A sixth-order partial differential governing equation for transverse free vibration is derived via variational principle with nonlocal elastic stress field theory. Analytical solutions for natural frequencies and transverse vibration modes are determined by applying a numerical analysis. Examples conclude that nonlocal stress effect tends to significantly increase stiffness and natural frequencies of a nanobeam. The relationship between natural frequency and nanoscale is also presented and its significance on stiffness enhancement with respect to the classical elasticity theory is discussed in detail. The effect of an initial axial tension, which also tends to enhance the nanobeam stiffness, is also concluded. The model and approach show potential extension to studies in carbon nanotube and the new result is useful for future comparison.

Keywords: cantilever nanobeam; free vibration; initial tension; nonlocal elasticity; nonlocal stress.

1. Introduction

With the rapid development of current technologies, miniaturized structures with nanoscale

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features can be precisely manufactured and applied in the so-called nano-electro-mechanical systems (NEMS) (Cagin *et al*. 1996, Gao and Zhao 2006, Jonsson *et al*. 2008). Carbon nanotubes and elastic beams with nanoscale thickness are most popular in these systems (Chen *et al*. 2008, Sato and Shima 2008, Unnikrishnan Zhao *et al*. 2008, Aluru 2008). Besides the load bearing capability, they can also be used as sensors and micrometers to detect the adsorption and to measure the interaction of certain molecules on their surface. This is because the adsorption and interaction may significantly alter the mechanical properties of the structures which will eventually lead to changes in mechanical behavior of the elements, including deformation and free vibration. Establishing an accurate model and relationship between load and vibration behaviour is thus a key issue for NEMS designs. Unfortunately, the classical elasticity theory fails to give such a relation because it lacks an intrinsic length scale and thus cannot capture the size-dependence vibration, as observed experimentally for nanoscale nanobeams. That is the reason why much attention has been paid to the analysis of classical structures with classical continuum mechanics, a subject of intensive research recently (Oz *et al*. 2001, Na *et al*. 2003, Wang *et al*. 2005, Parker and Orloske 2006).

There are molecular structures that can be modeled as nanobeams depending on geometry and configuration. One of them is a cantilever nanobeam and it is, in fact, one of the most important components in NEMS because it can be both a sensor as well as an actuator. Recently, there exists intensive research on dynamic behavior of nanobeams because of their potential prospects in NEMS or nano-machine components. Although nanobeams have found practical applications, analysis in this field has been lacking in particular the dynamics and vibration of pre-tensioned nanobeams.

The nonlocal elasticity theory was first developed by Eringen (1972, 1983) and Eringen and Edelen (1972) in the early 1970s. In recent years, the nonlocal stress theory and modeling for nanobeams have received increasing interest in nanomechanics research. This nonlocal continuum theory regards the stress at a point as a function of the strain states of all points in the body while the classical continuum mechanics assumes the stress state as only dependent uniquely on the strain state at that same point. This is in accordance with the atomic theory of lattice dynamics and experimental observations on phonon dispersion and so an internal size scale is introduced into the constitutive equations as a material parameter. In the limit when the effects of strains at other points are neglected, the nonlocal continuum theory reverts to the classical theory. The nonlocal elasticity theory has been applied in nanomechanics including lattice dispersion of elastic waves, wave propagation in composites, dislocation mechanics, fracture mechanics, surface tension fluids, etc. (Yu 1985, Yakobson *et al*. 1996, Reddy and Wang 1998, Shibutani *et al*. 1998, Mikkelsen and Tvergaard 1999, Sudak 2003, He *et al*. 2004, Zhang *et al*. 2005, Lu *et al*. 2006, Hu *et al*. 2008). In recent work, Lim and Wang (2007) Pintroduced an asymptotic representation of the one-dimensional nanobeam model via a variational principle approach and their nanobeam bending solutions based on nonlocal stress model were useful to engineers who designed micro- or nano-electromechanical devices. In another research by Tounsi *et al*. (2008), it was concluded that the scale coefficient was radius dependent.

In this paper, we attempt to investigate the nonlocal stress effects on a cantilever nanobeam with an axial tension and subsequently the study of its transverse free vibration. The model is described by a new sixth-order partial differential equation in dimensionless quantities via an exact variational principle approach. It is found that the presence of an initial tension and nonlocal stress do play significant roles in the free vibration behavior of a cantilever nanobeam in which the structural stiffness is greatly enhanced. The results are useful for designing nanoscale devices as components in NEMS.

2. Problem Definition and Modeling

Consider a uniform cantilever nanobeam with axial coordinate x, fixed at $x = 0$ and an initial axial tension N at the free end. The length of the nanobeam is L and the transverse deformation is w. The governing equation for transverse free vibration can be obtained by variational principle as follows.

For free vibration of a nanobeam without dissipation, energy changes from strain to kinetic forms and vibratory motion sustains at its natural frequency. The strain energy density u at an arbitrary point of a deformed nanobeam is given by (Lim 2008)

$$u = \frac{1}{2}E\varepsilon_{xx}^2 + \frac{1}{2}E\sum_{n=1}^{\infty}(-1)^{n+1}\tau^{2n}\left(\frac{d^n\varepsilon_{xx}}{d\bar{x}^n}\right)^2 + E\sum_{n=1}^{\infty}\left\{\tau^{2(n+1)}\sum_{m=1}^{n}\left[(-1)^{m+1}\frac{d^m\varepsilon_{xx}}{d\bar{x}^m}\frac{d^{(2(n+1)-m)}\varepsilon_{xx}}{d\bar{x}^{(2(n+1)-m)}}\right]\right\} \tag{1}$$

where $\bar{x} = \frac{x}{L}$ is the dimensionless coordinate, $\tau = \frac{e_0 a}{L}$ the nonlocal nanoscale parameter, E the Young's modulus, ε_{xx} the normal strain in axial direction and e_0 a constant dependent on each material, a an internal characteristic length (e.g. lattice parameter, C-C bond length, granular distance, etc.) (Eringen 1983). The total strain energy in the deformed body is

$$U = \int_V u\,dV \tag{2}$$

Following the variational principle, the variation of strain energy can be finally expressed as

$$\begin{aligned}\delta U = {} & \frac{EI}{L}\int_0^1\left[-\sum_{n=1}^{\infty}(2n-3)\tau^{2(n-1)}\frac{\partial^{2(n+1)}\overline{w}}{\partial\bar{x}^{2(n+1)}}\right]\delta\overline{w}\,d\bar{x} \\ & + \frac{EI}{L}\left[\sum_{n=1}^{\infty}(2n-3)\tau^{2(n-1)}\frac{\partial^{(2n+1)}\overline{w}}{\partial\bar{x}^{(2n+1)}}\delta\overline{w} - \sum_{n=1}^{\infty}(2n-3)\tau^{2(n-1)}\frac{\partial^{2n}\overline{w}}{\partial\bar{x}^{2n}}\frac{\partial\delta\overline{w}}{\partial\bar{x}}\right. \\ & + \sum_{n=1}^{\infty}(2n-1)\tau^{2n}\frac{\partial^{(2n+1)}\overline{w}}{\partial\bar{x}^{(2n+1)}}\frac{\partial^2\delta\overline{w}}{\partial\bar{x}^2} - \sum_{n=1}^{\infty}2n\tau^{2(n+1)}\frac{\partial^{2(n+1)}\overline{w}}{\partial\bar{x}^{2(n+1)}}\frac{\partial^3\delta\overline{w}}{\partial\bar{x}^3} + \sum_{n=1}^{\infty}(2n-1)\tau^{2(n+1)}\frac{\partial^{(2n+1)}\overline{w}}{\partial\bar{x}^{(2n+1)}}\frac{\partial^4\delta\overline{w}}{\partial\bar{x}^4} \\ & \left. - \sum_{n=1}^{\infty}(2n-1)\tau^{2(n+2)}\frac{\partial^{2(n+1)}\overline{w}}{\partial\bar{x}^{2(n+1)}}\frac{\partial^5\delta\overline{w}}{\partial\bar{x}^5} + \sum_{n=1}^{\infty}(2n-1)\tau^{2(n+3)}\frac{\partial^{(2n+3)}\overline{w}}{\partial\bar{x}^{(2n+3)}}\frac{\partial^6\delta\overline{w}}{\partial\bar{x}^6} + \cdots\right]\end{aligned} \tag{3}$$

where $\overline{w} = \frac{x}{L}$, $I = \iint_A y^2\,dA$ is the cross-sectional area moment of inertial and y is the transverse coordinate. In the presence an axial tension N at the free end, the work done is

$$V = \frac{N}{2}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2 dx = \frac{NL}{2}\int_0^1\left(\frac{\partial\overline{w}}{\partial\bar{x}}\right)^2 d\bar{x} \tag{4}$$

Variation of the work above is given by

$$\delta V = NL\int_0^1 \frac{\partial \bar{w}}{\partial \bar{x}}\frac{\partial \delta \bar{w}}{\partial \bar{x}}d\bar{x} = NL\left[\frac{\partial \bar{w}}{\partial \bar{x}}\delta \bar{w}\Big|_0^1 - \int_0^1 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\delta \bar{w}d\bar{x}\right] \tag{5}$$

For a nanobeam in free vibration, the kinetic energy due to transverse motion is

$$E_k = \frac{\rho}{2}\int_0^L \left(\frac{\partial w}{\partial t}\right)^2 dx = \frac{\rho L^3}{2T^2}\int_0^1 \left(\frac{\partial \bar{w}}{\partial \bar{t}}\right)^2 d\bar{x} \tag{6}$$

where $\bar{t} = \frac{t}{T}$ in which t, T are the time coordinate and period of vibration, respectively, and ρ is the density per unit length. Then, variation of the kinetic energy is given by

$$\delta E_k = \frac{\rho L^3}{T^2}\int_0^1 \frac{\partial \bar{w}}{\partial \bar{t}}\frac{\partial \delta \bar{w}}{\partial \bar{t}}d\bar{x} = \frac{\rho L^3}{T^2}\left[\frac{\partial \bar{w}}{\partial \bar{t}}\delta \bar{w}\Big|_0^1 - \int_0^1 \frac{\partial^2 \bar{w}}{\partial \bar{t}^2}\delta \bar{w}d\bar{x}\right] \tag{7}$$

For static equilibrium, the variational principal requires that

$$\delta(U - V - E_k) = 0 \tag{8}$$

which results in

$$\begin{aligned}
0 = & \int_0^1 \frac{EI}{L}\left[-\sum_{n=1}^{\infty}(2n-3)\tau^{2(n-1)}\frac{\partial^{2(n+1)}\bar{w}}{\partial \bar{x}^{2(n+1)}} + \frac{NL^2}{EI}\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + \frac{\rho L^4}{EIT^2}\frac{\partial^2 \bar{w}}{\partial \bar{t}^2}\right]\delta \bar{w}d\bar{x} \\
& + \frac{EI}{L}\left\{\left[\sum_{n=1}^{\infty}(2n-3)\tau^{2(n-1)}\frac{\partial^{(2n+1)}\bar{w}}{\partial \bar{x}^{(2n+1)}} - \frac{NL^2}{EI}\frac{\partial \bar{w}}{\partial \bar{x}} - \frac{\rho L^4}{EIT^2}\frac{\partial \bar{w}}{\partial \bar{t}}\right]\delta \bar{w}\right. \\
& - \sum_{n=1}^{\infty}(2n-3)\tau^{2(n-1)}\frac{\partial^{2n}\bar{w}}{\partial \bar{x}^{2n}}\frac{\partial \delta \bar{w}}{\partial \bar{x}} + \sum_{n=1}^{\infty}(2n-1)\tau^{2n}\frac{\partial^{(2n+1)}\bar{w}}{\partial \bar{x}^{(2n+1)}}\frac{\partial^2 \delta \bar{w}}{\partial \bar{x}^2} \\
& - \sum_{n=1}^{\infty}2n\tau^{2(n+1)}\frac{\partial^{2(n+1)}\bar{w}}{\partial \bar{x}^{2(n+1)}}\frac{\partial^3 \delta \bar{w}}{\partial \bar{x}^3} + \sum_{n=1}^{\infty}(2n-1)\tau^{2(n+1)}\frac{\partial^{(2n+1)}\bar{w}}{\partial \bar{x}^{(2n+1)}}\frac{\partial^4 \delta \bar{w}}{\partial \bar{x}^4} \\
& \left. - \sum_{n=1}^{\infty}(2n-1)\tau^{2(n+2)}\frac{\partial^{2(n+1)}\bar{w}}{\partial \bar{x}^{2(n+1)}}\frac{\partial^5 \delta \bar{w}}{\partial \bar{x}^5} + \sum_{n=1}^{\infty}(2n-1)\tau^{2(n+3)}\frac{\partial^{(2n+3)}\bar{w}}{\partial \bar{x}^{(2n+3)}}\frac{\partial^6 \delta \bar{w}}{\partial \bar{x}^6} + \cdots\right\}_0^1
\end{aligned} \tag{9}$$

Since $\delta\bar{w}$ cannot vanish, hence the governing equation of motion from Eq. (9) is

$$-\sum_{n=1}^{\infty}(2n-3)\tau^{2(n-1)}\frac{\partial^{2(n+1)}\bar{w}}{\partial \bar{x}^{2(n+1)}} + \bar{N}\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + \bar{\rho}\frac{\partial^2 \bar{w}}{\partial \bar{t}^2} = 0 \tag{10}$$

where $\bar{N} = \frac{NL^2}{EI}$ is the dimensionless axial tension, $\bar{\rho} = \frac{\rho L^4}{EIT^2}$ is the dimensionless density and the boundary conditions are obtained as

$$\left.\begin{array}{ll}
\sum_{n=1}^{\infty}(2n-3)\tau^{2(n-1)}\frac{\partial^{(2n+1)}\overline{w}}{\partial\bar{x}^{(2n+1)}}-\overline{N}\frac{\partial\overline{w}}{\partial\bar{x}}-\bar{\rho}\frac{\partial\overline{w}}{\partial\bar{t}}=0 & \text{or}\quad \overline{w}=0\\
\sum_{n=1}^{\infty}(2n-3)\tau^{2(n-1)}\frac{\partial^{2n}\overline{w}}{\partial\bar{x}^{2n}}=0 & \text{or}\quad \frac{\partial\overline{w}}{\partial\bar{x}}=0\\
\sum_{n=1}^{\infty}(2n-1)\tau^{2n}\frac{\partial^{(2n+1)}\overline{w}}{\partial\bar{x}^{(2n+1)}}=0 & \text{or}\quad \frac{\partial^{2}\overline{w}}{\partial\bar{x}^{2}}=0\\
\sum_{n=1}^{\infty}2n\tau^{2(n+1)}\frac{\partial^{2(n+1)}\overline{w}}{\partial\bar{x}^{2(n+1)}}=0 & \text{or}\quad \frac{\partial^{3}\overline{w}}{\partial\bar{x}^{3}}=0\\
\sum_{n=1}^{\infty}(2n-1)\tau^{2(n+1)}\frac{\partial^{(2n+1)}\overline{w}}{\partial\bar{x}^{(2n+1)}}=0 & \text{or}\quad \frac{\partial^{4}\overline{w}}{\partial\bar{x}^{4}}=0\\
\sum_{n=1}^{\infty}(2n-1)\tau^{2(n+2)}\frac{\partial^{2(n+1)}\overline{w}}{\partial\bar{x}^{2(n+1)}}=0 & \text{or}\quad \frac{\partial^{5}\overline{w}}{\partial\bar{x}^{5}}=0\\
\sum_{n=1}^{\infty}(2n-1)\tau^{2(n+3)}\frac{\partial^{(2n+3)}\overline{w}}{\partial\bar{x}^{(2n+3)}}=0 & \text{or}\quad \frac{\partial^{6}\overline{w}}{\partial\bar{x}^{6}}=0\\
\vdots & \text{or}\quad \vdots
\end{array}\right\}_{\bar{x}=0,1} \tag{11}$$

To investigate the nonlocal stress effect, the first nonlocal terms in Eqs. (10) and (11), which are the most important terms reflecting the nonlocal effects, are retained. The governing equation of motion with the most significant nonlocal terms are obtained as

$$-\tau^{2}\frac{\partial^{6}\overline{w}}{\partial\bar{x}^{6}}+\frac{\partial^{4}\overline{w}}{\partial\bar{x}^{4}}+\overline{N}\frac{\partial^{2}\overline{w}}{\partial\bar{x}^{2}}+\bar{\rho}\frac{\partial^{2}\overline{w}}{\partial\bar{t}^{2}}=0 \tag{12}$$

and the corresponding boundary conditions are

$$\left.\begin{array}{ll}
-\frac{\partial^{3}\overline{w}}{\partial\bar{x}^{3}}+\tau^{2}\frac{\partial^{5}\overline{w}}{\partial\bar{x}^{5}}-\overline{N}\frac{\partial\overline{w}}{\partial\bar{x}}-\bar{\rho}\frac{\partial\overline{w}}{\partial\bar{t}}=0 & \text{or}\quad \overline{w}=0\\
-\frac{\partial^{2}\overline{w}}{\partial\bar{x}^{2}}+\tau^{2}\frac{\partial^{4}\overline{w}}{\partial\bar{x}^{4}}=0 & \text{or}\quad \frac{\partial\overline{w}}{\partial\bar{x}}=0\\
\tau^{2}\frac{\partial^{3}\overline{w}}{\partial\bar{x}^{3}}+3\tau^{4}\frac{\partial^{5}\overline{w}}{\partial\bar{x}^{5}}=0 & \text{or}\quad \frac{\partial^{2}\overline{w}}{\partial\bar{x}^{2}}=0
\end{array}\right\}_{\text{at }\bar{x}=0,1} \tag{13}$$

For linear free vibration of a nanobeam, the vibration modes are harmonic in time. From Eq. (12), time-dependent transverse deformation of nanobeam can be represented by

$$\overline{w}(\bar{x},\bar{t})=\overline{W}_n(\bar{x})e^{i\bar{\omega}_n\bar{t}} \tag{14}$$

where $\overline{W}_n(\bar{x})$ is the dimensionless vibration amplitude, $n = 1, 2, 3, \mathrm{K}$ denotes the vibration mode number and $\overline{\omega}_n$ is the dimensionless natural frequency. Substituting Eq. (14) into Eq. (12), the equation of motion becomes

$$\tau^2\frac{d^6\overline{W}_n}{d\bar{x}^6}-\frac{d^4\overline{W}_n}{d\bar{x}^4}-\overline{N}\frac{d^2\overline{W}_n}{d\bar{x}^2}+\bar{\rho}\overline{\omega}_n^2\overline{W}_n = 0 \tag{15}$$

Further substituting Eq. (14) into Eq. (13), the boundary conditions become

$$\left.\begin{array}{lll} -\dfrac{d^3\overline{W}_n}{d\bar{x}^3}+\tau^2\dfrac{d^5\overline{W}_n}{d\bar{x}^5}-\overline{N}\dfrac{d\overline{W}_n}{d\bar{x}}-i\overline{\omega}_n\bar{\rho}\overline{W}_n = 0 & \text{or} & \overline{W}_n = 0 \\ -\dfrac{d^2\overline{W}_n}{d\bar{x}^2}+\tau^2\dfrac{d^4\overline{W}_n}{d\bar{x}^4} = 0 & \text{or} & \dfrac{d\overline{W}_n}{d\bar{x}} = 0 \\ \tau^2\dfrac{d^3\overline{W}_n}{d\bar{x}^3}+3\tau^4\dfrac{d^5\overline{W}_n}{d\bar{x}^5} = 0 & \text{or} & \dfrac{d^2\overline{W}_n}{d\bar{x}^2} = 0 \end{array}\right\}_{\text{at } \bar{x} = 0,\,1} \tag{16}$$

For free vibration, the deflection of a nanobeam can be represented by

$$\overline{W}_n(\bar{x}) = C_n e^{i\beta_{jn}\bar{x}} \tag{17}$$

where C_n as an arbitrary nonzero constant. Substituting this expression into Eq. (15), we obtain a dispersion relation as

$$\tau^2\beta_n^6+\beta_n^4-\overline{N}\beta_n^2-\bar{\rho}\overline{\omega}_n^2 = 0 \tag{18}$$

Since Eq. (18) is a sixth-order polynomial in terms of β_n, the six roots are denoted by $\beta_{jn}(j = 1, 2, K, 6)$, respectively. Because only linear free vibration is concerned, the superposition of the six solutions with respect to each root β_{jn} is also a solution of the governing Eq. (15). Hence

$$\overline{W}_n(\bar{x}) = C_{1n}e^{i\beta_{1n}\bar{x}}+C_{2n}e^{i\beta_{2n}\bar{x}}+C_{3n}e^{i\beta_{3n}\bar{x}}+C_{4n}e^{i\beta_{4n}\bar{x}}+C_{5n}e^{i\beta_{5n}\bar{x}}+C_{6n}e^{i\beta_{6n}\bar{x}} \tag{19}$$

where $C_{jn}(j = 1, 2, K, 6)$ are six constants of integration associated with Eq. (15) which is a sixth-order ordinary differential equation.

3. Numerical Examples and Effects on Natural Frequencies

To illustrate the effect of nanoscale parameter and initial axial tension on the transverse vibration of a nanobeam, the following numerical example for a cantilever nanobeam is presented and discussed in detail.

In defining boundary conditions of nonlocal beam models, the consistent expressions such as bending moment or shear force should be given by the nonlocal forms but not their classical counterparts with $\tau = 0$. It is likely to be overlooked in some of the applications (Zhang *et al.* 2005), where the classical expressions for the boundary bending moments were still used. In the present work, for a cantilever nanobeam fixed at $\bar{x} = 0$ and free at $\bar{x} = 1$, the dimensionless nonlocal boundary conditions from Eq. (16) are given by

$$\overline{W}_n(0) = 0; \quad \frac{d\overline{W}_n(0)}{d\bar{x}} = 0; \quad \tau^2 \frac{d^3\overline{W}_n(0)}{d\bar{x}^3} + 3\tau^4 \frac{d^5\overline{W}_n(0)}{d\bar{x}^5} = 0;$$
$$-\frac{d^3\overline{W}_n(1)}{d\bar{x}^3} + \tau^2 \frac{d^5\overline{W}_n(1)}{d\bar{x}^5} - \overline{N}\frac{d\overline{W}_n(1)}{d\bar{x}} - i\overline{\omega}_n\bar{\rho}\overline{W}_n(1) = 0;$$
$$-\frac{d^2\overline{W}_n(1)}{d\bar{x}^2} + \tau^2 \frac{d^4\overline{W}_n(1)}{d\bar{x}^4} = 0; \quad \frac{d^2\overline{W}_n(1)}{d\bar{x}^2} = 0 \tag{20}$$

Substituting Eq. (19) into Eq. (20), the equations can be expressed in a matrix form as

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \beta_{1n} & \beta_{2n} & \beta_{3n} & \beta_{4n} & \beta_{5n} & \beta_{6n} \\ \beta_{1n}^3 - 3\tau^2\beta_{1n}^5 & \beta_{2n}^3 - 3\tau^2\beta_{2n}^5 & \beta_{3n}^3 - 3\tau^2\beta_{3n}^5 & \beta_{4n}^3 - 3\tau^2\beta_{4n}^5 & \beta_{5n}^3 - 3\tau^2\beta_{5n}^5 & \beta_{6n}^3 - 3\tau^2\beta_{6n}^5 \\ \beta_{1n}^2 e^{i\beta_{1n}} & \beta_{2n}^2 e^{i\beta_{2n}} & \beta_{3n}^2 e^{i\beta_{3n}} & \beta_{4n}^2 e^{i\beta_{4n}} & \beta_{5n}^2 e^{i\beta_{5n}} & \beta_{6n}^2 e^{i\beta_{6n}} \\ e^{i\beta_{1n}}(\beta_{1n}^3 + \tau^2\beta_{1n}^5 - \overline{N}\beta_{1n} - \overline{\omega}_n\bar{\rho}) & e^{i\beta_{2n}}(\beta_{2n}^3 + \tau^2\beta_{2n}^5 - \overline{N}\beta_{2n} - \overline{\omega}_n\bar{\rho}) & e^{i\beta_{3n}}(\beta_{3n}^3 + \tau^2\beta_{3n}^5 - \overline{N}\beta_{3n} - \overline{\omega}_n\bar{\rho}) & e^{i\beta_{4n}}(\beta_{4n}^3 + \tau^2\beta_{4n}^5 - \overline{N}\beta_{4n} - \overline{\omega}_n\bar{\rho}) & e^{i\beta_{5n}}(\beta_{5n}^3 + \tau^2\beta_{5n}^5 - \overline{N}\beta_{5n} - \overline{\omega}_n\bar{\rho}) & e^{i\beta_{6n}}(\beta_{6n}^3 + \tau^2\beta_{6n}^5 - \overline{N}\beta_{6n} - \overline{\omega}_n\bar{\rho}) \\ \beta_{1n}^4 e^{i\beta_{1n}} & \beta_{2n}^4 e^{i\beta_{2n}} & \beta_{3n}^4 e^{i\beta_{3n}} & \beta_{4n}^4 e^{i\beta_{4n}} & \beta_{5n}^4 e^{i\beta_{5n}} & \beta_{6n}^4 e^{i\beta_{6n}} \end{pmatrix} \begin{pmatrix} C_{1n} \\ C_{2n} \\ C_{3n} \\ C_{4n} \\ C_{5n} \\ C_{6n} \end{pmatrix} = 0 \tag{21}$$

For nontrivial solutions, the determinant of matrix (21) must be zero, or

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \beta_{1n} & \beta_{2n} & \beta_{3n} & \beta_{4n} & \beta_{5n} & \beta_{6n} \\ \beta_{1n}^3 - 3\tau^2\beta_{1n}^5 & \beta_{2n}^3 - 3\tau^2\beta_{2n}^5 & \beta_{3n}^3 - 3\tau^2\beta_{3n}^5 & \beta_{4n}^3 - 3\tau^2\beta_{4n}^5 & \beta_{5n}^3 - 3\tau^2\beta_{5n}^5 & \beta_{6n}^3 - 3\tau^2\beta_{6n}^5 \\ \beta_{1n}^2 e^{i\beta_{1n}} & \beta_{2n}^2 e^{i\beta_{2n}} & \beta_{3n}^2 e^{i\beta_{3n}} & \beta_{4n}^2 e^{i\beta_{4n}} & \beta_{5n}^2 e^{i\beta_{5n}} & \beta_{6n}^2 e^{i\beta_{6n}} \\ e^{i\beta_{1n}}(\beta_{1n}^3 + \tau^2\beta_{1n}^5 - \overline{N}\beta_{1n} - \overline{\omega}_n\bar{\rho}) & e^{i\beta_{2n}}(\beta_{2n}^3 + \tau^2\beta_{2n}^5 - \overline{N}\beta_{2n} - \overline{\omega}_n\bar{\rho}) & e^{i\beta_{3n}}(\beta_{3n}^3 + \tau^2\beta_{3n}^5 - \overline{N}\beta_{3n} - \overline{\omega}_n\bar{\rho}) & e^{i\beta_{4n}}(\beta_{4n}^3 + \tau^2\beta_{4n}^5 - \overline{N}\beta_{4n} - \overline{\omega}_n\bar{\rho}) & e^{i\beta_{5n}}(\beta_{5n}^3 + \tau^2\beta_{5n}^5 - \overline{N}\beta_{5n} - \overline{\omega}_n\bar{\rho}) & e^{i\beta_{6n}}(\beta_{6n}^3 + \tau^2\beta_{6n}^5 - \overline{N}\beta_{6n} - \overline{\omega}_n\bar{\rho}) \\ \beta_{1n}^4 e^{i\beta_{1n}} & \beta_{2n}^4 e^{i\beta_{2n}} & \beta_{3n}^4 e^{i\beta_{3n}} & \beta_{4n}^4 e^{i\beta_{4n}} & \beta_{5n}^4 e^{i\beta_{5n}} & \beta_{6n}^4 e^{i\beta_{6n}} \end{vmatrix} = 0 \tag{22}$$

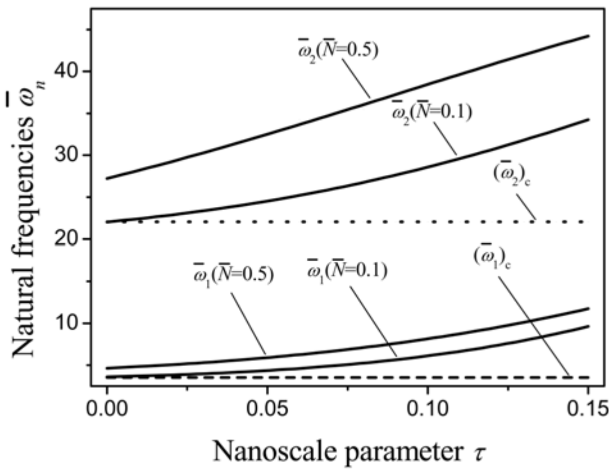

Fig. 1 Effects of nanoscale and initial tension on the first two natural frequencies for $\bar{\rho} = 0.01$

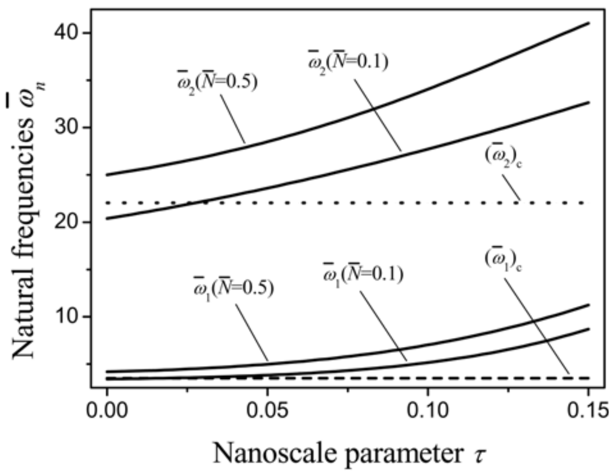

Fig. 2 Effects of nanoscale and initial tension on the first two natural frequencies for $\bar{\rho} = 0.03$

By combining Eqs. (18) and (22), the seven unknown quantities $\beta_{jn}(j = 1, 2, 3, 4, 5, 6)$ and $\overline{\omega}_n$ can be solved. Subsequently, substituting the results into Eqs. (21), the analytical solutions of *n*th vibration mode $\overline{W}_n$ in Eq. (19) and transverse deformation $\overline{w}$ in Eq. (14) can be solved to the extent of an arbitrary constant, for instance $C_{1n} \neq 0$.

For comparison with the classical elasticity theory, the natural frequency for free vibration of a classical cantilever beam without initial tension is (Liu *et al.* 1998)

$$(\omega_n)_c = \lambda_n^2 \sqrt{\frac{EI}{\rho}}, \qquad n = 1, 2, 3 \cdots \tag{23}$$

where $(\omega_n)_c$ for $n = 1, 2, 3 \cdots$ is the physical vibration frequencies from the classical vibration theory and λ_n can be obtained from the following transcendental equation

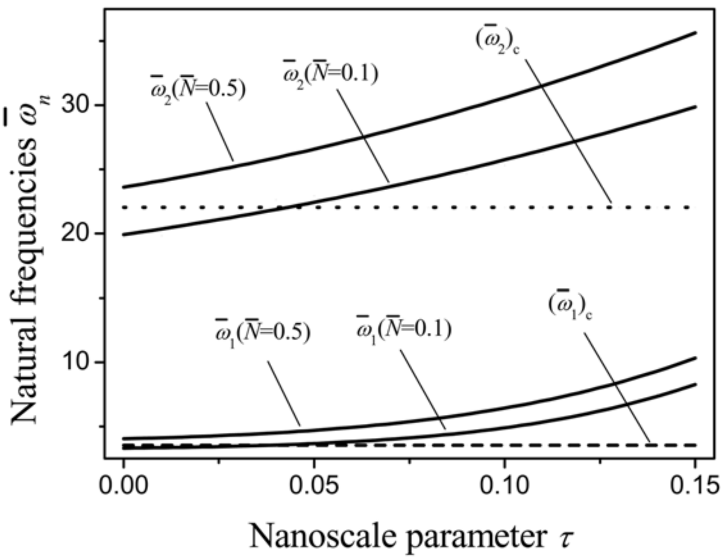

Fig. 3 Effects of nanoscale and initial tension on the first two natural frequencies for $\bar{\rho} = 0.05$

$$\cos(\lambda_n L)\cosh(\lambda_n L) + 1 = 0 \tag{24}$$

where L is the length of the classical cantilever beam. Solving Eq. (24), the first two dimensionless natural frequencies are approximately

$$(\bar{\omega}_1)_c = \lambda_1^2 L^2 = (\omega_1)_c L^2 \sqrt{\frac{\rho}{EI}} = 1.875^2;\ (\bar{\omega}_2)_c = \lambda_2^2 L^2 = (\omega_2)_c L^2 \sqrt{\frac{\rho}{EI}} = 4.694^2 \quad \text{(25a,b)}$$

Effects of nanoscale parameter and initial tension on the first two natural frequencies are shown in Fig. 1 for $\bar{\rho} = 0.01$. The classical solutions in Eq. (25a,b) are also presented and compared. Similarly, the relationships for $\bar{\rho} = 0.03$ and $\bar{\rho} = 0.05$ are illustrated in Figs. 2 and 3, respectively, to indicate the nonlocal effects as well as varying dimensionless density on the first two natural frequencies.

It is observed that the nanoscale and initial axial tension affect the natural frequencies very significantly. The nonlocal effect enhances nanobeam stiffness and thus causes higher frequencies as compared with the classical solutions. Specifically, the first two natural frequencies increase with increasing τ, which indicates that stronger nonlocal effects cause higher natural frequencies. Similarly, it is also observed that stronger initial axial tension induces higher nanobeam stiffness and thus higher vibration frequency. In general, with certain increase in τ or N, the first two vibration frequencies presented in Figs. 1 to 3 could be more than double of their original values.

In these three figures, the classical vibration solutions $(\bar{\omega}_1)_c$ and $(\bar{\omega}_2)_c$, as defined in Eq. (25), assume identical values because these are dimensionless parameters are not only independent of nonlocal effects but also invariant for varying density. However, for a cantilever nanobeam with nonlocal effects, it is obvious that a larger density ρ leads to a lower frequency by comparing Figs.1, 2 and 3. This observation shows the unique features of vibration characteristics which are size-dependent and not noticeable in classical vibration theory.

4. Conclusions

In this paper, the free transverse vibration of a cantilever nanobeam with initial axial tension is solved based on a new nonlocal stress field theory. A high-order partial differential equation which governs the vibration behavior is obtained via the variational principle. Applying a numerical method, the effects of nanoscale parameter, initial tension, as well as the dimensionless density on natural frequencies are investigated in detail. It is found that the nanoscale parameter and initial tension induce higher frequencies as compared with the classical beam solutions while the dimensionless density results in lower frequencies. Stiffness of a nanobeam is greatly enhanced by the presence of a nanoscale as well as an initial axial tension. In summary, unique and significant, size-dependent vibration characteristics not present in classical vibration theory are noted in nonlocal nanobeam vibration.

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