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Solution for a semi-infinite plate with radial crack and radial crack emanating from circular hole under bi-axial loading by body force method

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Abstract. Machine or structural members subjected to fatigue loading will have a crack initiated during early part of their life. Therefore analysis of members with cracks and other discontinuities is very important. Finite element method has enjoyed widespread use in engineering, but it is not convenient for crack problems as the region very close to crack tip is to be discretized with very fine mesh. However, as the body force method (BFM), requires only the boundary of the discontinuity (crack or hole) to be discretized it is easy versatile technique to analyze such problems. In the present work fundamental solution for concentrated load $x + iy$ acting in the semi-infinite plate at an arbitrary point $z_0 = x_0 + iy_0$ is considered. These fundamental solutions are in complex form $\phi(z)$ and $\psi(z)$ (England 1971). These potentials are known as Melan potentials (Ramakrishna 1994). A crack in the semi-infinite plate as shown in Fig. 1 is considered. This crack is divided into number of divisions. By applying pair of body forces on a division, the resultant forces on the remaining 'N'divisions are to be found for which $\phi_1(z)$ and $\psi_1(z)$ are derived. Body force method is applied to calculate stress intensity factor for crack in semi-infinite plate. Also for the case of crack emanating from circular hole in semi-infinite plate radial stress, hoop stress and shear stress are calculated around the hole and crack. Convergent results are obtained by body force method. These results are compared with FEM results.

Keywords: body force method; complex potentials; Melan potentials.

1. Introduction

Stress analysis of components is important in design and development of machines and structures. For determining stress distribution in a body, fourth order partial differential equation (Bi-harmonic equation) for the stress function is to be solved. Closed form solutions are available for a simple geometry, load and boundary conditions. Presence of geometrical discontinuity is a big challenge. A

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slight change in the shape and geometrical discontinuity will affect the problem entirely different. Timoshenko and Goodier (1982), Love (1934) and Muskhelishvili (1953) have developed methods to solve a number of problems of practical relevance. Complex variable approach is well suited for stress analysis of problem and discontinuity with holes, cracks etc. Honein and Hermann (1988) employed complex variable approach to solve the problem of infinite plate with circular hole / inclusions subjected to in plane concentrated load. Ramakrishna (1994) extended the approach to solve the problem of half plane. This problem becomes very difficult when the geometry of discontinuity becomes complex. The boundary (body) force method proposed by H.Nisitani which is hybrid theoretical and numerical method is found to be versatile method when discontinuities like holes or cracks are present.

2. Body force method

This method is based on the fundamental solution of concentrated load acting in either infinite plate or semi-infinite plate. To start with infinite / semi-infinite plate is considered with hole. The actual hole is considered to be imaginary circle. The imaginary circle is divided into number of equal divisions. At the mid-point of each divisions of circle unit concentrated load in x and y directions are applied. The resultant forces due to all concentrated loads at all divisions are calculated from the complex potentials of fundamental solution of finite / semi-infinite plate. These resultant forces acting on each divisions of imaginary circle are nullified due to bi-axial load. Finally a square matrix with influence coefficients (resultant force at N^{th} division due to concentrated load acting at M^{th} division), column matrix with unknown body forces and column matrix containing resultant forces on all divisions due to bi-axial load is obtained. After solving this, body forces in x-direction and y-directions are obtained. If these body forces are applied at mid-point of each division, the imaginary circle boundary becomes stress free and it will be equivalent to hole in a plate subjected to bi-axial load. The stress at any point is given by the sum of the stresses due to all body forces and bi-axial loading (principle of super position).

If it is a case of crack in a infinite / semi-infinite plate, actual crack is considered to be an imaginary ellipse with major axis equal to the length of crack and minor axis approaching zero. The imaginary crack is divided into number of equal divisions. At the mid-point of each divisions of crack a pair of body forces are applied. The resultant forces due to pair of body forces at all divisions are calculated from the complex potentials of fundamental solution of finite / semi-infinite plate. Fundamental solution for pair of loads applied on crack are obtained by differentiating fundamental solutions of concentrated loads applied on infinite / semi-infinite plates.

3. Crack in semi-infinite plate

Consider semi-infinite plate with crack subjected to bi-axial load as shown in Fig. 1. The stress intensity factor is to be determined at both ends of crack. (At "A" and "B") by body force method. Crack is treated as an infinitely slender elliptical hole, i.e., the short axis approaches zero. On the imaginary crack, continuous distribution pairs of symmetrical body forces acting at infinitesimal distances are applied. When a semi- infinite plate is subjected to a concentrated force $X + iY$ acting at an arbitrary point $z_0 = x_0 + iy_0$, as shown in Fig. 3, in the plane stress condition, the Melan

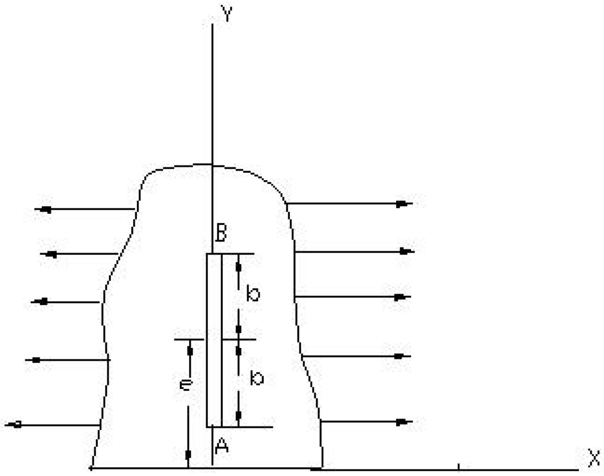

Fig. 1 Crack in semi-infinite plate

potentials are semiexpressed as (Ramakrishna 1994, England 1971, Wang 1994).

$$\phi(z) = -\frac{(X+iY)}{8\pi}[\log(z-z_{01})+\kappa\log(z-\bar{z}_{01})]+\frac{(X-iY)}{8\pi}\frac{(z-z_{01})}{(z-\bar{z}_{01})} \tag{1}$$

$$AA = \frac{(X-iY)}{8\pi}[A+B+C+D+E] \tag{2}$$

$$\psi(z) = \frac{(X+iY)}{8\pi}\left[\frac{\bar{z}_{01}}{(z-z_{01})}+\frac{\kappa z}{(z-\bar{z}_{01})}\right]+AA \tag{3}$$

where

$$A = \kappa\log(z-z_{01})$$

$$B = \frac{zz_{01}}{(z-\bar{z}_{01})^2}$$

$$C = \log(z-\bar{z}_{01})$$

$$D = \frac{z}{(z-\bar{z}_{01})}$$

$$E = \frac{z^2}{(z-\bar{z}_{01})^2}$$

The components of the stress and the resultant force can be expressed in terms of the complex potentials as (England 1971).

$$\sigma_x+\sigma_y = 2\lfloor\phi'(z)+\bar{\phi}'(z)\rfloor \tag{4}$$

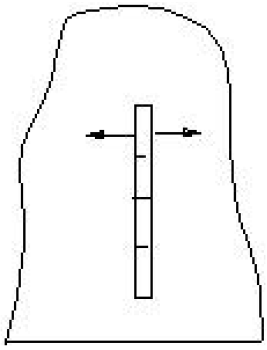

Fig. 2 Pair of body forces on a division of crack

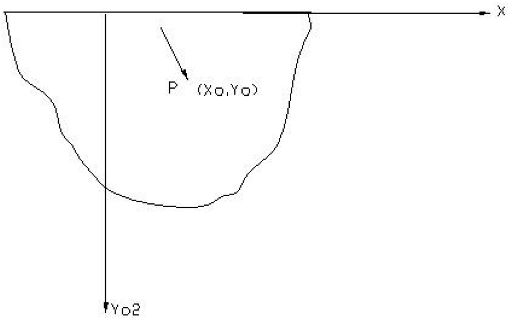

Fig. 3 Melan-type loading

$$\sigma_x - \sigma_y + 2i\tau_{xy} = -2[z\bar{\phi}''(z) + \bar{\psi}'(z)] \tag{5}$$

$$p_x + ip_y = -i[\phi(z) + z\bar{\phi}'(z) + \bar{\psi}(z)]_A^B \tag{6}$$

where P_x and P_y are resultant force along divided segments in x and y directions respectively.

When a pair of body forces acts at an arbitrary site on the imaginary crack, as shown in Fig. 2, one can obtain the resultant force along the path from point "A" to point "B" (A and B are the end points of each divisions of crack) through the following procedure. First, put $X = T$, $Y = 0$ in Eqs. (1) and (3) Second find derivatives of $\phi(z)$ and $\psi(z)$ with respect to 'x_0' (Fig. 3) and multiply each of them by "ε". Then, the complex potentials of a pair of body forces can be obtained and denoted by $\phi_1(z)$ and $\psi_1(z)$

$$\phi_1(z) = -\frac{R}{8\pi}\left[-\frac{1}{(z-z_0)} - \frac{2}{(z-\bar{z}_0)} - \frac{(z-z_0)}{(z-\bar{z}_0)^2}\right] \tag{7}$$

$$\psi_1(z) = \frac{R}{8\pi}\left[\begin{array}{l} -\dfrac{2}{(z-z_0)} + \dfrac{\bar{z}_0}{(z-z_0)^2} + \dfrac{z}{(z-\bar{z}_0)^2} - \dfrac{2zz_0}{(z-\bar{z}_0)^3} \\ -\dfrac{1}{(z-\bar{z}_0)} + \dfrac{2z^2}{(z-\bar{z}_0)^3} \end{array}\right] \tag{8}$$

Differentiating Eq. (6) twice, we get

$$\phi_1^1(z) = -\frac{R}{8\pi}\left[\frac{1}{(z-z_o)^2} + \frac{1}{(z-\bar{z}_o)^2} + \frac{2(z-z_0)}{(z-\bar{z}_o)^3}\right] \tag{9}$$

$$\phi_1^{11}(z) = -\frac{R}{8\pi}\left[-\frac{2}{(z-z_o)^3} - \frac{6(z-z_o)}{(z-\bar{z}_o)^4}\right] \tag{10}$$

Differentiating Eq. (7)

$$\psi_1^1(z) = \frac{R}{8\pi}\left[\begin{array}{l}\frac{2}{(z-z_o)^2} - \frac{2\bar{z}_0}{(z-z_o)^3} - \frac{2z}{(z-\bar{z}_o)^3} - \frac{6zz_0}{(z-\bar{z}_o)^4} \\ + \frac{1}{(z-\bar{z}_o)^2} - \frac{6z^2}{(z-\bar{z}_o)^4}\end{array}\right] \tag{11}$$

where $R = T\varepsilon$ (product of 'T' and 'ε') and 'ε'is twice the modulus of x_0 ($\varepsilon = 2\,|x_0| \rightarrow 0$) which tends to approach zero. 'T' is pair of body force shown in Fig. 2. This 'R' is substituted for $dR = 4\gamma(\mathrm{x})\sqrt{(a^2-x^2)}dx$ (Wang 1994, Nisitani 1978).

3.1 DETAILED PROCEDURE

The imaginary crack is divided into 'N' equal segments as shown in Fig. 4. The values of the density functions of pairs of boundary forces of the dividing points and the ends of the imaginary crack are denoted by $\gamma_1, \gamma_2, \gamma_3 \ldots\ldots\ldots \gamma_{n+1}$, assuming that they change linearly within each segment. The values of density functions of pairs of boundary forces are the unknowns, whose total number is $N+1$. By using the above equations, the resultant force along each segment created by the applied pairs of boundary forces should compensate that created by bi-axial loading. Thus,'N' equations can be established by boundary conditions of the semi-infinite plate with one crack (Wang 1994). But the number of equations is less than the number of unknowns. In order to increase the number of equations, other segments $K_1K_2, K_2K_3, K_3K_4\ldots\ldots$ are taken, considering the boundary conditions, where $K_2, K_3, K_4\ldots\ldots$ are the midpoints of the previously divided segments of

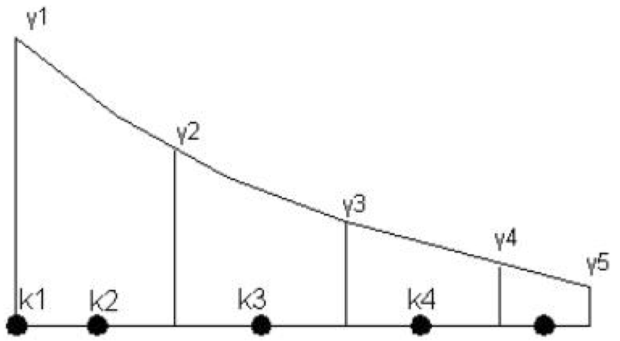

Fig. 4 Crack divisions

imaginary crack. Then the number of unknowns is equal to the number of equations. The solution of this problem can be uniquely determined. "When continuous distribution pairs of boundary forces are applied on the full imaginary crack length. i.e., from point (-a, 0) to (a,0),(-a and +a are extreme ends of the crack), these pairs of boundary forces acting on the element dx, denoted by dR can be expressed as $dR = 4\gamma(\mathrm{x})\ \sqrt{(a^2 - x^2)}dx$ (Wang 1994, Nisitani 1978) where x is the x-coordinate of the element dx. Here, $\gamma(x)$ is called the density function of pairs of boundary forces". Substituting dR for R in Eqs. (6), (7), (8), (9) and (10) and integrating along the full imaginary crack, $[P_x]_A^B$ and $[P_y]_A^B$ are obtained which are created by the continuous distribution pairs of boundary forces acting on the full imaginary crack length. Because the density function of pairs of boundary forces is unknown function, which is determined by numerical method, the integrals along the full imaginary crack length are replaced by summations of the integrals along each segment of the imaginary crack.

3.2 Numerical results

The stress intensity factor s at A and B of Fig. 1 can be calculated as $K_{1A} = \gamma_1\sqrt{b}$ and $K_{1B} = \gamma_N\sqrt{b}$ where γ_1 and γ_N are the density functions at first and N^{th} divisions of crack respectively. The stress intensity correction factors obtained by body force method are compared with FEM results as shown in below Table 1 and Table 2.

Stress intensity correction factor at 'A' is given by

$F_{1A} = \dfrac{\gamma_1\sqrt{b}}{\sigma\sqrt{b}}$ where 'σ' is applied bi-axial stress.

4. Solution for a semi-infinite plate with radial crack emanating from circular hole under bi-axial loading by body force method

A semi-infinite plate with crack emanating from circular hole is shown in Fig. 5.

Table 1 Stress intensity correction factor

b/e	BFM	FEM
0.2	1.0112	1.0114
0.4	1.0528	1.0428
0.6	1.1491	1.1462
0.8	1.3879	1.3719

Table 2 Stress intensity correction factor

DIVISIONS OF CRACK	BFM	FEM
12	1.09033	1.09112
16	1.09063	1.09022
24	1.09091	1.08021
32	1.09104	1.08022
48	1.09115	1.09111

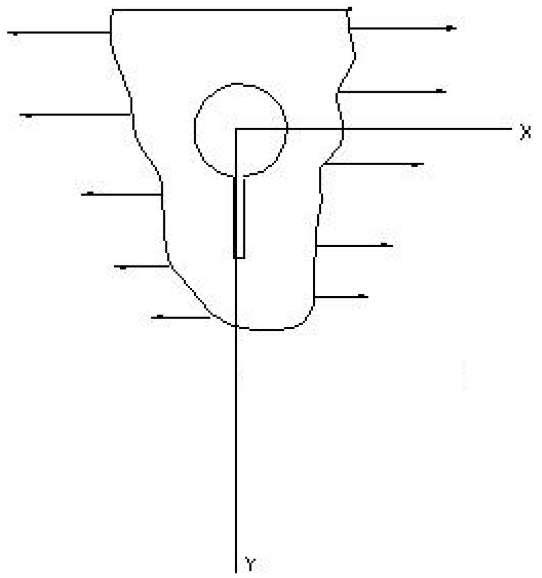

Fig. 5 Crack and hole

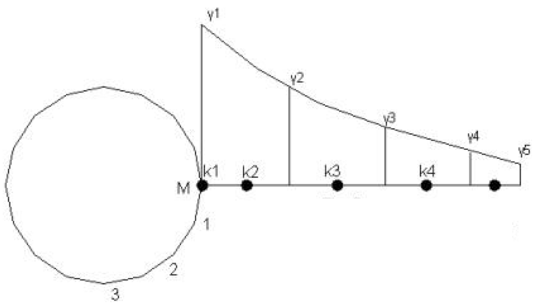

Fig. 6 Crack and hole divisions

The periphery of the imaginary circular hole is divided into 'N' equal segments, as shown in Fig. 6. At the midpoint of each segment, a x-direction and a y-direction concentrated boundary force are applied. The imaginary crack is divided into 'M' equal segments as shown in Fig. 6. The values of the density functions of pairs of boundary forces of the dividing points and the ends of the imaginary crack are denoted by $\gamma_1, \gamma_2, \gamma_3 \ldots\ldots\ldots \gamma_{n+1}$ assuming that they change linearly within each segment. These concentrated forces and the values of density functions of pairs of boundary forces are the unknowns, whose total number is $2N + M + 1$. By using the above equations, the resultant force along each segment created by the applied concentrated boundary forces and the applied pairs of boundary forces should compensate that created by bi-axial loading. Thus, $2N + M$ equations can be established by boundary conditions of the semi- infinite plate with one hole and one crack whose divided case is shown in Fig. 5. But the number of equations is less than the number of unknowns. In order to increase the number of equations, other segments K_1K_2, $K_2K_3, K_3K_4\ldots\ldots$ are taken, considering the boundary conditions, where $K_2, K_3, K_4\ldots\ldots$ are the midpoints of the previously divided segments.

Imaginary crack. Then the number of unknowns is equal to the number of equations. These equations are expressed in matrix form. The solution of this problem can be uniquely determined. For 32 divisions of hole and 4 divisions of crack, we get 69 × 70 as order of matrix as shown below.

$$\begin{bmatrix} P_{X1}^{X1}\ldots\ldots P_{X1}^{X32} & P_{X1}^{Y1}\ldots\ldots P_{X1}^{Y32} & P_{X1}^{M1}\ldots\ldots P_{X1}^{M5} \\ P_{Y1}^{X1}\ldots\ldots P_{Y1}^{X32} & P_{Y1}^{Y1}\ldots\ldots P_{Y1}^{Y32} & P_{Y1}^{M1}\ldots\ldots P_{Y1}^{M5} \\ . & & \\ .P_{Y32}^{X1}\ldots\ldots P_{Y32}^{X32} & P_{Y32}^{Y1}\ldots\ldots P_{Y32}^{Y32} & P_{Y32}^{M1}\ldots\ldots P_{Y32}^{M5} \\ . & & \\ .P_{M1}^{X1} \qquad P_{M1}^{X32} & P_{M1}^{Y1}\ldots\ldots P_{M1}^{Y32} & P_{M1}^{M1}\ldots\ldots P_{M1}^{M5} \\ . & & \\ . & & \\ P_{M5}^{X1} \qquad P_{M5}^{X32} & P_{M5}^{Y1}\ldots\ldots P_{M5}^{Y32} & P_{M5}^{M1}\ldots\ldots P_{M5}^{M5} \end{bmatrix} \begin{bmatrix} \rho_{X1} \\ . \\ .\rho_{X32} \\ .\rho_{Y1} \\ . \\ \rho_{Y32} \\ \gamma_1 \\ . \\ \gamma_5 \end{bmatrix} = \begin{bmatrix} P_{X1}^{B} \\ . \\ P_{32}^{B} \\ P_{Y1}^{B} \\ . \\ P_{Y32}^{B} \\ P_{M1}^{B} \\ . \\ P_{M5}^{B} \end{bmatrix} \tag{12}$$

Influence coefficients inside the square matrix are resultant forces along each hole divisions and crack divisions due to concentrated unit loads along x and y directions and pair of body forces where

P_{X1}^{X2} is X-component of resultant force acting along 1st division of hole due to unit concentrated load acting along x-direction in 2nd division of hole.

P_{X1}^{M1} is X-component of resultant force acting along 1st division of hole due to pair of body forces acting in $M1$ division of crack.

P_{X1}^{Y2} is X-component of resultant force acting along 1st division of hole due to unit concentrated load acting along y-direction in 2nd division of hole.

P_{Y1}^{M1} is resultant force acting along 1st division of hole in y-direction due to pair of body forces acting in $M1$ division of crack.

P_{X1}^{B} is X-component of resultant force acting along 1st division of hole due to Bi-axial load.

P_{Y1}^{B} is Y-component of resultant force acting along 1st division of hole due to Bi-axial load.

ρ_{X1} is body force in x-direction.

ρ_{Y1} is body force in y-direction.

4.1 Numerical results

The circular hole of radius 2 units is considered in the problem with 4 mm radial crack subjected to bi-axial type loading of 10 units. The normalized radial, hoop and shear stresses along radial direction at angle 10 degree clock-wise from x-axis (Fig. 5) is computed and compared with FEM results. Stresses along 10-degree radial line are shown in Fig. 7, Fig. 8, and Fig. 9. Convergence study is carried out by making hole divisions 8, 16, 32, 64 and Crack divisions 4, 6, 8, 10. Convergent results of radial stress, hoop stress and shear stresses at a distance of 4 mm from the origin along 10degree radial line from x-axis (Fig. 5) is shown in Table3, 4 and 5.

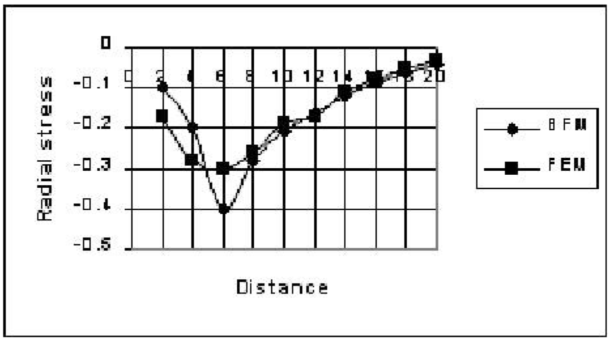

Fig. 7 Variation of normalised radial stress

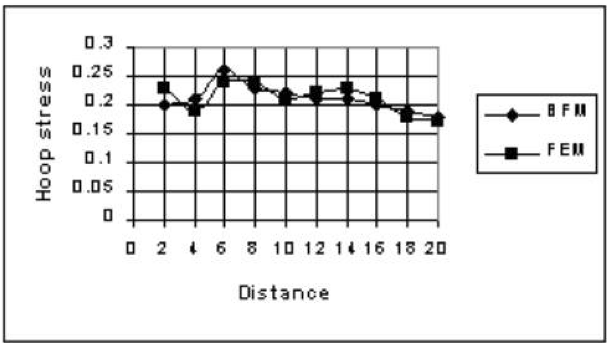

Fig. 8 Variation of normalised hoop stress

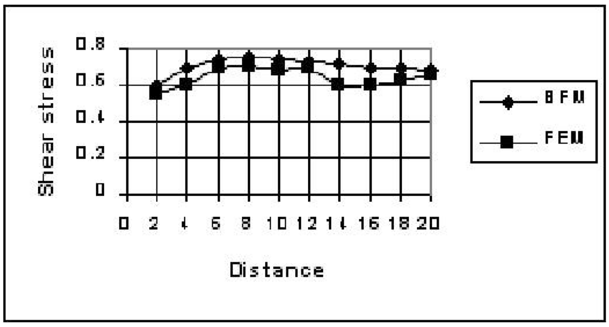

Fig. 9 Variation of normalised shear stress

Table 3 Convergent results of radial stress

HOLE DIVISIONS	CRACK DIVISIONS	BFM	FEM
8	4	-.2546	-.2513
16	6	-.2544	
32	8	-.2541	
64	10	-.2543	

Table 4 Convergent results of hoop stress

HOLE DIVISIONS	CRACK DIVISIONS	BFM	FEM
8	4	.1593	.1588
16	6	.1596	
32	8	.1598	
64	10	.1591	

Table 5 Convergent results of shear stress

HOLE DIVISIONS	CRACK DIVISIONS	BFM	FEM
8	4	.6123	.6131
16	6	.6128	
32	8	.6121	
64	10	.6126	

5. Conclusions

As the BFM is semi-numerical method and requires less mesh divisions than FEM, it can be concluded that BFM gives better results for crack problems. The body force method is applied for the case of hole in infinite plate subjected in plane concentrated load (Kelvin-type loading) (Manjunath and Ramakrishna 2007c), flamant case with hole (Manjunath and Ramakrishna 2006) and Melan case with hole (Manjunath and Ramakrishna 2007a). The results obtained are compared with analytical solution (involution technique) (Ramakrishna 1994) and FEM solutions. It is found that FEM solutions are not as close as body force method with involution technique. As body force method is semi-numerical, it is very powerful technique to handle crack problems

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Nomenclature

a: Half crack length
BFM: Body Force Method
ε: Crack thickness
γ: Angle of inclination of load w.r.t x-axis
κ: (3--4ν) for plane strain $(3 - \nu) / (1 + \nu)$ for plane stress
Z: $X + iY$
$\gamma_1\ \gamma_2\ \gamma_3$ …. Density functions

$\phi(Z)$, $\psi(Z)$: Complex potentials, functions of
Complex variable $X + iY$
ρ_{X1}: Body force in X-direction
ρ_{Y1}: Body force in Y-direction
z_0: $x_0 + iy_0$
$\bar{z}$: Conjugate of $z = (x - iY)$
$\bar{z}_{01}$: Conjugate of $z_{01} = x_{01} - iY_{01}$