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Computational modelling for description of rubber-like materials with permanent deformation under cyclic loading

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Abstract. When carbon-filled rubber specimens are subjected to cyclic loading, they do not return to their initial state after loading and subsequent unloading, but exhibit a residual strain or permanent deformation. We propose a specific form of the pseudo-elastic energy function to represent cyclic loading for incompressible, isotropic materials with stress softening and residual strain. The essence of the pseudo-elasticity theory is that material behaviour in the primary loading path is described by a common elastic strain energy function, and in unloading, reloading or secondary unloading paths by a different strain energy function. The switch between strain energy functions is controlled by the incorporation of a damage variable into the strain energy function. An extra term is added to describe the permanent deformation. The finite element implementation of the proposed model is presented in this paper. All parameters in the proposed model and elastic law can be easily estimated based on experimental data. The numerical analyses show that the results are in good agreement with experimental data.

Keywords: Mullins effect; pseudo-elastic model; finite element method; permenant deformation.

1. Introduction

Rubber-like materials exhibit a strongly non-linear behaviour characterised by large strains and a non-linear stress-strain response. When a rubber specimen is subjected to cyclic loading, stress-softening phenomena are also observed (Mullins 1947, Mullins & Tobin 1957, Harwood *et al*. 1967, Beatty & Krishnaswamy 2000, Krishnaswamy & Beatty 2000, Guo 2006, Guo & Sluys 2006, Govindjee & Simó 1992a, 1992b). This stress-softening phenomenon was observed in a thorough experimental study of carbon-black filled rubber vulcanizates by Mullins (1947) and has subsequently become widely known as the Mullins effect. Moreover, another important phenomenon for carbon-filled rubber is that after loading and subsequent unloading rubber specimens, in general, do not return to their initial state, but exhibit a residual deformation. Different approaches have been developed to deal with this phenomenon by many researchers. Representative works include Lion (1996), Septanika (1998), Miehe and Keck (2000), Drozdov and Dorfmann (2001), Besdo and Ihlemann 2003 and Dorfmann and Ogden 2004. Both experimental data and constitutive models can be found in the above-cited works.

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Ogden & Roxburgh (1999) have proposed a theory of pseudo-elasticity to describe the damage-induced stress-softening effect in rubber-like solids. Furthermore, Dorfmann and Ogden (2003) apply this theory to the hysteretic cycles associated with partial unloading and reloading following loading after appropriate pre-conditioning aimed at eliminating the Mullins effect. Recently, Dorfmann and Ogden (2000) used pseudo-elasticity to capture Mullins effect and residual strain effects with the inclusion of two variables in the energy function. But, this constitutive model cannot describe the evolution of rubber softening and permanent deformation and has too many adjustable parameters and functions, which can, to some degree, only be determined arbitrarily (2005).

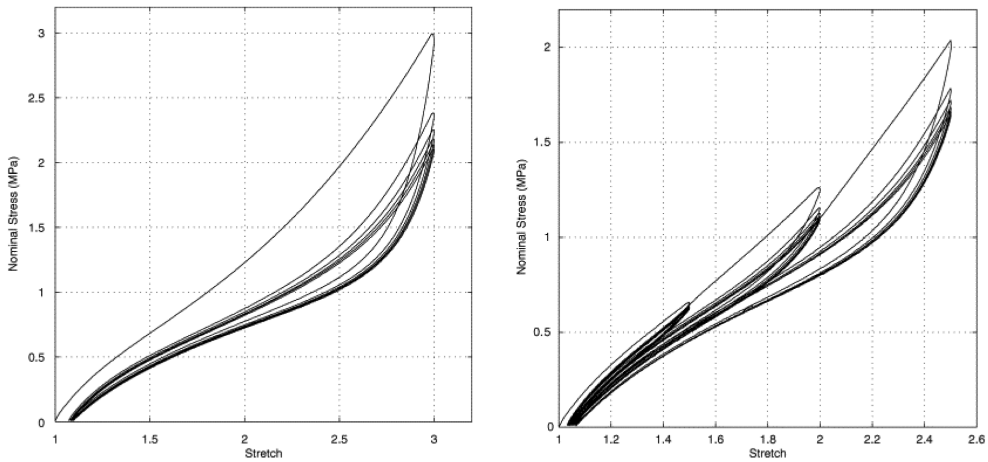
In this paper, attention is restricted to the development of constitutive models for the computational analysis of the static behaviour including strong non-linearity, Mullins effect and residual strain due to a strain history. Firstly, the theory of pseudo-elasticity is used to construct the constitutive equation for describing the inelastic effects of carbon black filled rubber under cyclic loading. The constitutive elastic laws and necessary derivation of finite element implementation are also presented in this section. Numerical analyses show the characteristics of the proposed model.

2. Experimental observation

It is necessary to repeat the main experimental observations of non-linearity, the Mullins effect and permanent deformation in order to construct a phenomenological model allowing the representation of these phenomena (Mullins 1947, Lion 1996, 1997, Dorfmann and Ogden 2004). Dorfmann and Ogden (2004) carried out experimental results at constant temperature under cyclic loading-unloading uniaxial tension. The specimens have the shape of a dumbbell and contain different amounts of carbon black. One set of results is illustrated in Fig. 1. Fig. 1(a) gives experimental results with 20 phr (by volume) of Carbon Black filler with maximum stretch $\lambda = 3.0$. Fig. 1(b) shows the experimental results with the same Carbon Black filler but with maximum stretches of $\lambda = 1.5$, $\lambda = 2.0$ and $\lambda = 2.5$. The nominal stress was determined as the ratio of axial force to the undeformed cross-sectional area of a specimen (2 mm × 4 mm) in the stress free state. These experiments are characterized by:

- Most of the Mullins effect for both unfilled and CB-filled rubber occurs during the first loading cycle. After several loading-unloading cycles (e.g. six cycles), the stress-strain responses are essentially repeatable and additional stress is negligible.
- The magnitude of stress softening during the first few loading-unloading cycles depends on the value of the maximum strain achieved.
- In general a CB-filled rubber after loading and subsequent unloading does not return to its initial state corresponding to the natural stress-free configuration. The main part of the residual deformation is generated during the first loading-unloading cycle. After several loading-unloading cycles, the residual deformation appears to reach a fixed value.
- The accumulated residual deformation depends on the value of the maximum strain during the previous loading cycle. The magnitude of the accumulated residual deformation does not depend linearly on the maximum strain.
- All these phenomena depend on the proportion of carbon black in the compound. In particular, both the stress softening effect and residual deformation increase with increasing filler content.

Experimental results reveal that both the Mullins effect and residual deformation depend on the maximum value of strain during the previous loading history.



(a) (b)

Fig. 1 Nominal stress-stretch curves of periodic uniaxial extension tests of a particle-reinforced specimen with 20 phr of carbon black (a) with maximum stretch $\lambda = 3.0$ and (b) with maximum stretches of $\lambda = 1.5$, $\lambda = 2.0$ and $\lambda = 2.5$

3. Constitutive model

We propose a specific form of the pseudo-elastic energy function to represent cyclic loading for incompressible, isotropic material with stress softening and residual strain. Similar to the ideal stress-softening case treated in our former paper (Guo & Sluys 2006), the essence of the pseudo-elasticity theory is that material behaviour in the primary loading path is described by a common elastic strain energy function $W(\mathbf{F})$, and in unloading, reloading or secondary unloading paths by a different strain energy function. An extra term is added to describe permanent deformation. The pseudo-elastic energy function has the following form

$$W(\mathbf{F}, \eta) = \eta\, W_0(\mathbf{F}) + f(\eta)\, W_r(\mathbf{F}, \mathbf{F}_m) + \phi(\eta) \tag{1}$$

where $\mathbf{F}$ is the deformation gradient tensor for an undamaged elastic materials, $\mathbf{F}_m$ is the deformation gradient tensor at the maximum deformation in the loading history, $W_0(\mathbf{F})$ is the strain energy function for an undamaged elastic materials and $W_r(\mathbf{F})$ is strain energy related to pemanent deformation. The second term in the right hand of Eq. (1) is related to the phenomenon of residual strains, which depend on the strain history. $\phi(\eta)$ is referred to as a dissipation function.

From the point of initiation of unloading on, the damage variable η is active. It is taken to be dependent on the deformation gradient (the damage evolves with deformation) and following Ogden and Roxburgh (1999) this dependence can be expressed as

$$\frac{\partial W(\mathbf{F}, \eta)}{\partial \eta} = W_0(\mathbf{F}) + f'(\eta) W_r(\mathbf{F}, \mathbf{F}_m) + \phi'(\eta) = 0 \tag{2}$$

The second Piola-Kirchhoff stress is then given by

$$\boldsymbol{\tau} = 2\frac{\partial W(\mathbf{F}, \eta)}{\partial \mathbf{C}} = 2\frac{\partial W(\mathbf{F}, \eta)}{\partial \mathbf{C}} + 2\frac{\partial W(\mathbf{F}, \eta)}{\partial \eta}\frac{\partial \eta}{\partial \mathbf{C}} \tag{3}$$

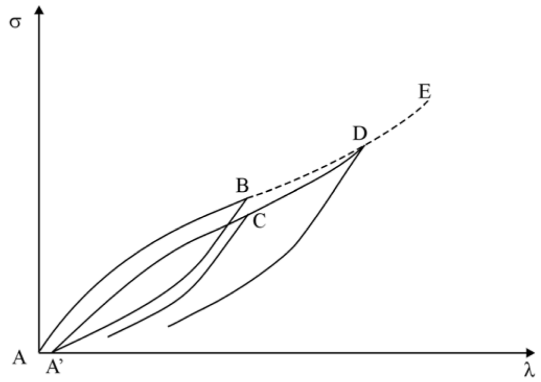


Fig. 2 A schematic uniaxial stress-stretch response with stress softening and permanent deformation

In which **C** is the right Cauchy-Green stretch tensor. Considering the constraint of Eq. (2), Eq. (3) becomes

$$\boldsymbol{\tau} = 2\frac{\partial W(\mathbf{F}, \eta)}{\partial \mathbf{C}} \tag{4}$$

Eq. (1) is intended to simulate different branches of loading, unloading, reloading and secondary unloading (Fig. 2) by means of a different expression of η.

3.1 Primary loading branch

The form of the pseudo-elastic energy function Eq. (1) should be reduced to the standard forms of energy function W_0 when we use it to describe the mechanical behaviour of isotropic materials on the primary loading path (curve ABDE in Fig. 2). The following constraints in Eq. (1) are necessary

$$\eta = 1, f(1) = 0, \ \phi(1) = 0 \qquad \text{on primary loading path} \tag{5}$$

Therefore, for primary loading from the natural (stress free) configuration, the energy function Eq. (1) becomes

$$W(\mathbf{F}, 1) = W_0(\mathbf{F}) \tag{6}$$

The second Piola-Kirchhoff stress can be calculated by differentiation of the energy function (Eq. (6)) with respect to the right Cauchy-Green stretch tensor

$$\boldsymbol{\tau}_0 = 2\frac{\partial W(\mathbf{F}, 1)}{\partial \mathbf{C}} = 2\frac{\partial W_0(\mathbf{F})}{\partial \mathbf{C}} \tag{7}$$

The incremental stress-strain relation is obtained by differentiation of Eq. (7) with respect to the right Cauchy-Green stretch tensor, yielding

$$\mathbf{D} = 4\frac{\partial^2 W_0}{\partial \mathbf{C}^2} \tag{8}$$

3.2 Unloading branch

When unloading (curve BA' in Fig. 2) is initiated from any point on the primary loading path, the variable η becomes active and the constraints in Eq. (5) no longer hold. We take a specific form of the pseudo-elastic energy function of Eq. (1) to represent unloading behaviour. Substituting Eq. (1) into Eq. (4), the second Piola-Kirchhoff stress can be obtained

$$\boldsymbol{\tau} = 2\eta\frac{\partial W_0(\mathbf{F})}{\partial \mathbf{C}} + 2f(\eta)\frac{\partial W_r(\mathbf{F}, \mathbf{F}_m)}{\partial \mathbf{C}} = \eta\boldsymbol{\tau}_0 + f(\eta)\boldsymbol{\tau}_r \tag{9}$$

In which $\boldsymbol{\tau}_0$ represent stress without influence of damage (stress softening) and $\boldsymbol{\tau}_r$ represent stress influenced by permanent deformation. The first term on the right hand side of Eq. (9) is the main effect of stress softening, therefore it is clear that we should have $0 < \eta \le 1$ on the unloading path and associate unloading with decreasing η.

When unloading reaches to the stress-free state, residual strain occurs. This is equivalent to the stress becoming negative when the strain (supposed as tensile strain) returns to zero. The first term in Eq. (9) disappears and η is defined as η_m when the strain is zero. The constant η_m is the minimum value of damage parameter. If we take

$$f(\eta_m) = 1 \tag{10}$$

then, $\boldsymbol{\tau}_r$ is related to the residual stress in the original configuration

$$\boldsymbol{\tau}_r = 2\frac{\partial W_r(\mathbf{F}, \mathbf{F}_m)}{\partial \mathbf{C}} \tag{11}$$

The residual strain depends on the maximum value of strain during the previous loading history, so that, $\boldsymbol{\tau}_r$ should be a function of maximum value of strain and depending only on maximum strain in the history and does not change with the current state of deformation **F** or **C**. Therefore

$$\boldsymbol{\tau}_r = \boldsymbol{\tau}_r(\mathbf{C}_m) \tag{12}$$

The magnitude of the accumulated residual strain does not depend linearly on the maximum strain. Obviously, the residual stress has a similar character. If the maximum value of strain is a tensile strain, the corresponding residual stress will be negative, and if the maximum value of strain is a compression strain, the corresponding residual stress will be positive. For convenience of implementation in the finite element method, here we use W_r(C, C_r) instead of W_r(F, F_r) and define

$$W_r(\mathbf{C}, \mathbf{C}_m) = -\sum_{i=1,3} (K_1 * (C_{iim} - 1)/\sqrt{ABS(C_{iim} - 1)} * C_{ii}) \tag{13}$$

where K_1 is a material parameter and C_{iim} are the components of the Right Cauchy-Green stretch tensor at state of maximum strain during the previous loading history. Then $\boldsymbol{\tau}_r$ becomes

$$\boldsymbol{\tau}_r = 2\begin{pmatrix} -K_1 * (C_{11m} - 1)/\sqrt{ABS(C_{11m} - 1)} \\ -K_1 * (C_{22m} - 1)/\sqrt{ABS(C_{22m} - 1)} \\ -K_1 * (C_{33m} - 1)/\sqrt{ABS(C_{33m} - 1)} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{14}$$

When unloading initiates from the loading path of simple tension the value of C_{11m} is larger than

1 and C_{22m} as well as C_{33m} are smaller than 1. Based on Eq. (14) the residual stress $\boldsymbol{\tau}_{r11}$ is negative and $\boldsymbol{\tau}_{r22}$ as well as $\boldsymbol{\tau}_{r33}$ are positive. These results are consistent to the physical phenomenon of simple tension with permanent deformation.

Eqs. (5), (9) and (10) show that function $f(\eta)$ leads to a residual stress separate from the total stress. Simplifying this separation, we assume $f(\eta)$ to be directly proportional to η and takes the form,

$$f(\eta) = \frac{1-\eta}{1-\eta_m} \tag{15}$$

This definition ensures $f(1) \equiv 0$ on the loading path, in which $\eta = 1$, and $f(\eta_m) \equiv 1$ when the strain returns to origin. The damage parameter η can be defined in terms of the deformation gradient. Considering that η should satisfy $0 < \eta \leq 1$ and decreases when unloading evolves, η is defined as

$$\eta = 1 - \frac{1}{r} erf\left(\frac{W_{\max} - W_0(\mathbf{F})}{m(W_{\max} - W_{00})}\right) \tag{16}$$

where m and r are positive parameters (material constants) and *erf*( ) is the error function. $W_0(\mathbf{F})$ is the strain energy corresponding to the same deformation $\mathbf{F}$ for an undamaged elastic material. W_{00} is the strain energy at the origin in the stress free state. For most forms of strain energy functions, the value of W_{00} is zero. $W_{\max}$ is the value of strain energy at the point on the primary loading path from which unloading is initiated ($\eta = 1$). Obviously, the strain energy $W_{\max}$ is the current maximum value of the energy achieved on the loading path. In accordance with the properties of $W_0(\mathbf{F})$, W_m increases along the primary loading path. We emphasize that, in general, the value of η derived from Eq. (16) will depend on the values of the deformation attained on the primary loading path and the specific formulation of $W_0(\mathbf{F})$.

When the material returns to the origin from primary loading and no deformation remains, η attains its minimum value η_m.

$$\eta_m = 1 - \frac{1}{r} erf\left(\frac{1}{m}\right) \tag{17}$$

Differentiation of Eq. (9) and consideration of Eq. (11) yield the incremental stress-strain relation

$$D = 4\eta\frac{\partial^2 W_0}{\partial \mathbf{C}^2} + 4\frac{\partial W_0}{\partial \mathbf{C}}\left(\frac{\partial \eta}{\partial \mathbf{C}}\right)^T - \frac{4}{1-\eta_m}\frac{\partial W_r}{\partial \mathbf{C}}\left(\frac{\partial \eta}{\partial \mathbf{C}}\right)^T \tag{18}$$

in which

$$\frac{\partial \eta}{\partial \mathbf{C}} = \frac{\partial}{\partial z}\left(1 - \frac{1}{r} erf(z)\right)\frac{\partial}{\partial \mathbf{C}} = \frac{1}{rm}\frac{2}{\sqrt{\pi}} e^{-\left(\frac{1}{m}\left(\frac{W_m - W_0}{W_{\max} - W_{00}}\right)\right)^2}\left(\frac{\partial W_0}{\partial \mathbf{C}}\right)^T \tag{19}$$

with

$$z = \frac{1}{m}\left(\frac{W_m - W_0}{W_{\max} - W_{00}}\right) \tag{20}$$

3.3 Reloading branch

The value of η is monotonously decreasing from its initial value 1 to its minimum value η_m

during the unloading evolution. We assume that at a specific value of $W_0(\mathbf{F})$, written as W_{mr}, the material is again subjected to loading. The corresponding value of η is η_{mr}, which is kept constant during the reloading process (curve A'CD in Fig. 2). The form of the pseudo-energy function for reloading still remains the form of Eq. (1), but η is changed to η_r according to

$$W(\mathbf{C}, \eta_r) = \eta_r W_0(\mathbf{C}) + f(\eta_r) W_r(\mathbf{C}, \mathbf{C}_m) + \phi(\eta_r) \tag{21}$$

where the variable η_r will increase from η_{mr} on. The value for η_{mr} could be equal to η_m or larger than η_m depending on the starting point of reloading. When reloading reaches to the point, at which unloading is initiated, the relevant value of $W_0(\mathbf{C})$ is equal to the value of $W_{\max}$, which is the maximum energy in the strain history. But, the value of η_r may not return to the value 1 and the reloading path may not rejoin the primary loading path in this point. This is consistent with experimental results. Finally, η_r can reach the value 1 and return to the undamaged path if reloading continues after energy exceeds $W_{\max}$ by a certain amount. To fulfil these requirements, a suitable expression of a monotonic increasing function to be used for η_r

$$\eta_r = \eta_{mr} + (1 - \eta_{mr}) erf\left(\frac{1}{m_1}\left(\frac{W_0(\mathbf{C}) - W_{mr}}{(W_{\max} - W_{00})}\right)^{r_1}\right) \tag{22}$$

where c_1 and m_1 are material constants. This equation satisfies that η_r is equal to η_{mr} when reloading commences and η_r may return to value 1 when the value of $W_0(\mathbf{C})$ becomes large enough.

The second Piola-Kirchhoff stress can be obtained from Eqs. (4) and (21)

$$\boldsymbol{\tau} = 2\eta_r \frac{\partial W_0(\mathbf{C})}{\partial \mathbf{C}} + 2f(\eta_r)\frac{\partial W_r(\mathbf{C}, \mathbf{C}_m)}{\partial \mathbf{C}} = \eta_r \boldsymbol{\tau}_0 + f(\eta_r)\boldsymbol{\tau}_r \tag{23}$$

in which

$$f(\eta_r) = \frac{1 - \eta_r}{1 - \eta_m} \tag{24}$$

Differentiation of Eq. (23) and consideration of Eq. (11) yield the incremental stress-strain relation

$$D = 4\eta_r \frac{\partial^2 W_0}{\partial \mathbf{C}^2} + 4\frac{\partial W_0}{\partial \mathbf{C}}\left(\frac{\partial \eta_r}{\partial \mathbf{C}}\right)^T - \frac{4}{1 - \eta_m}\frac{\partial W_r}{\partial \mathbf{C}}\left(\frac{\partial \eta_r}{\partial \mathbf{C}}\right)^T \tag{25}$$

where

$$\frac{\partial \eta_r}{\partial \mathbf{C}} = \frac{2(1 - \eta_{mu}) r_1 (W_0 - W_{mu})^{r_1 - 1}}{m_1 \sqrt{\pi}(W_m - W_{00})^{r_1}} e^{-z_1^2}\left(\frac{\partial W_0}{\partial \mathbf{C}}\right) \tag{26}$$

with

$$z_1 = \frac{1}{m_1}\left(\frac{W_0 - W_{mu}}{W_m - W_{00}}\right)^{r_1} \tag{27}$$

3.4 Secondary unloading branch

Secondary unloading is defined when unloading commences from the reloading path. Unloading from the primary loading path is considered as primary unloading, in short, we call it unloading. Unloading after reloading can be initiated from two different locations. One supposes that the material response on the reloading path returns to the primary loading path (D is joint point in Fig. 2) corresponding to $\eta_r = 1$. The other possibility is that the material response on the reloading path remains in a damaged state and parameter η_r is smaller than 1. In both cases, η_{mu} is assumed to be equal to η_r and W_{mu} be the value of energy at the starting point (point C in Fig. 2) of the (secondary) unloading process. η_u replaces η in the pseudo-elastic energy function of Eq. (8.1) to describe secondary unloading,

$$W(\mathbf{F}, \eta_u) = \eta_u W_0(\mathbf{F}) + f(\eta_u) W_r(\mathbf{F}, \mathbf{F}_m) + \phi(\eta_u) \tag{28}$$

where the variable η_u will decrease from η_{mu}, which may be either equal to 1 or less than 1 and related to unloading or secondary unloading, respectively. We select the variable η_u so that

$$\eta_u = \eta_{mu}\left(1 - \frac{1}{r} erf\left(\frac{W_{mu} - W_0}{m(W_{\max} - W_{00})}\right)\right) \tag{29}$$

If $\eta_{mu} = 1$, reloading returns to the primary loading path, consequently, secondary unloading becomes unloading and Eq. (29) returns to the form of Eq. (16). The second Piola-Kirchhoff stress can be calculated from Eqs. (4) and (28)

$$\boldsymbol{\tau} = 2\eta_u \frac{\partial W_0(\mathbf{F})}{\partial \mathbf{C}} + 2f(\eta_u)\frac{\partial W_r(\mathbf{F}, \mathbf{F}_m)}{\partial \mathbf{C}} = \eta_u \boldsymbol{\tau}_0 + f(\eta_u)\boldsymbol{\tau}_r \tag{30}$$

in which $f(\eta)$ becomes

$$f(\eta_r) = \frac{1 - \eta_r}{1 - \eta_m} \tag{31}$$

The incremental stress-strain relation

$$\mathbf{D} = 4\eta_u \frac{\partial^2 W_0}{\partial \mathbf{C}^2} + 4\frac{\partial W_0}{\partial \mathbf{C}}\left(\frac{\partial \eta_u}{\partial \mathbf{C}}\right)^T - \frac{4}{1 - \eta_m}\frac{\partial W_r}{\partial \mathbf{C}}\left(\frac{\partial \eta_u}{\partial \mathbf{C}}\right)^T \tag{32}$$

with

$$\frac{\partial \eta}{\partial \mathbf{C}} = \frac{\partial}{\partial z}\left(\eta_u\left(1 - \frac{1}{r} erf(z)\right)\right)\frac{\partial z}{\partial \mathbf{C}} = \frac{2\eta_u}{rm(W_{\max} - W_{00})\sqrt{\pi}} e^{-\left(\frac{1}{m}\left(\frac{W_m - W_0}{W_{\max} - W_{00}}\right)\right)^2}\left(\frac{\partial W_0}{\partial \mathbf{C}}\right)^T \tag{33}$$

3.5 Summary of the formula

The formula describing the evolution of cyclic loading can be summarized as follows. The pseudo-elastic energy function reads

$$W(\mathbf{F}, \eta) = \eta W_0(\mathbf{F}) + f(\eta) W_r(\mathbf{F}, \mathbf{F}_m) + \phi(\eta) \tag{34}$$

in which

$$W_r(\mathbf{C}, \mathbf{C}_m) = -\sum_{i=1,3}\left(K_1(C_{iim} - 1)/\sqrt{ABS(C_{iim} - 1)}C_{ii}\right) \tag{35}$$

and

$$f(\eta) = \frac{1 - \eta}{1 - \eta_m} \tag{36}$$

with

$$\eta = \begin{cases} 1 & \text{Primary loading} \\ 1 - \dfrac{1}{r} erf\left(\dfrac{W_{\max} - W_0}{m(W_{\max} - W_{00})}\right) & \text{Unloading} \\ \eta_{mr} + (1 - \eta_{mr}) erf\left(\dfrac{1}{m_1}\left(\dfrac{W_0 - W_{mr}}{W_{\max} - W_{00}}\right)^{r_1}\right) & \text{Reloading} \\ \eta_{mu}\left(1 - \dfrac{1}{r} erf\left(\dfrac{W_{mu} - W_0}{m(W_{\max} - W_{00})}\right)\right) & \text{Secondary unloading} \end{cases} \tag{37}$$

The second Piola-Kirchhoff stress

$$\boldsymbol{\tau} = 2\eta \frac{\partial W_0(\mathbf{F})}{\partial \mathbf{C}} + 2f(\eta) \frac{\partial W_r(\mathbf{F}, \mathbf{F}_m)}{\partial \mathbf{C}} = \eta \boldsymbol{\tau}_0 + f(\eta) \boldsymbol{\tau}_r \tag{38}$$

The incremental stress-strain relation

$$\mathbf{D} = 4\eta \frac{\partial^2 W_0}{\partial \mathbf{C}^2} + 4 \frac{\partial W_0}{\partial \mathbf{C}} \cdot \left(\frac{\partial \eta}{\partial \mathbf{C}}\right)^T + f'(\eta) \frac{\partial W_r}{\partial \mathbf{C}} \cdot \left(\frac{\partial \eta}{\partial \mathbf{C}}\right)^T \tag{39}$$

4. Numerical analysis

For the numerical analyses, the elastic strain energy proposed by Gao's (1997) has been used:

$$W = a(I_1^n + I_{-1}^n) \tag{40}$$

where a and n are positive material parameters, I_1, and I_{-1} are strain invariants (Guo & Sluys 2006b).

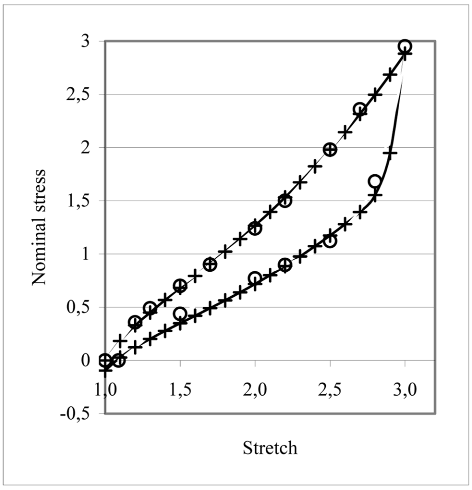
Substitution of Eq. (40) into Eq. (38) and Eq. (39) yields the necessary implementation formula of the stress-strain relation $\boldsymbol{\tau}(\mathbf{C}, \eta)$ and the incremental stress-strain relation $\mathbf{D}$. For a detailed description of the algorithmic aspects the readers is refered to former publications (Guo 2006, Guo & Sluys 2006a, 2006b).

We now apply this model to simulate the combination of stress-softening and residual strain accumulation in particle-reinforced rubber and compare the numerical results with experimental data of Fig. 1, which have been carried out by Dorfmann and Ogden (2004).

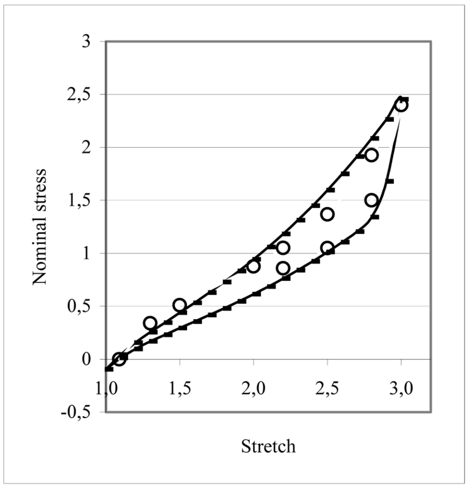
Table 1 Estimated values of material parameters for Pseudo-elastic model

Experiment	Material parameters	Values
Cyclic uniaxial tension	a	0.0457
	n	1.72
	r	3.2
	m	0.38
	K_1	0.013
	r_1	0.35
	m_1	1.2

In the numerical calculation, primary loading is fully determined by the strain energy in Eq. (40). The model parameters a and n are estimated based on experimental data of primary loading in Fig. 1a. The model parameter K_1 is obtained by extending the unloading path until the strain returns to zero, where Eqs. (12) and (17) are activated. Then, the parameters r and m may be determined based on the unloading data and the parameters r_1 and m_1 are based on reloading data. These values of parameters are summarized in Table 1.



(a)



(b)

Fig. 3 Comparison of stress-stretch curves between numerical and experimental data of uniaxial tension: (a) primary loading and unloading, (b) reloading and secondary unloading. (■) experimental data, (○) numerical results

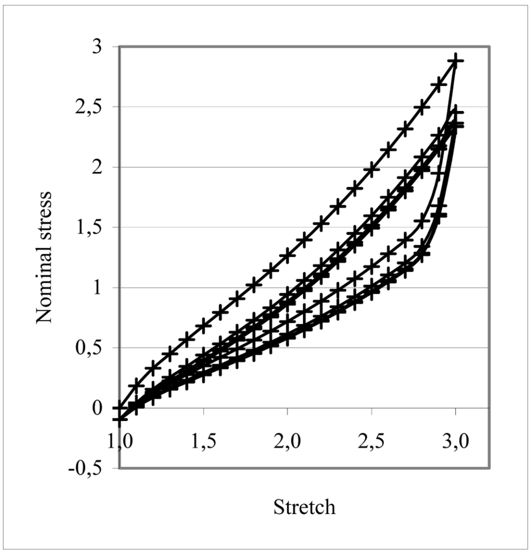


Fig. 4 Numerical simulation of uniaxial tension under cyclic loading with maximum stretch $\lambda = 3.0$: nominal stress-stretch curves

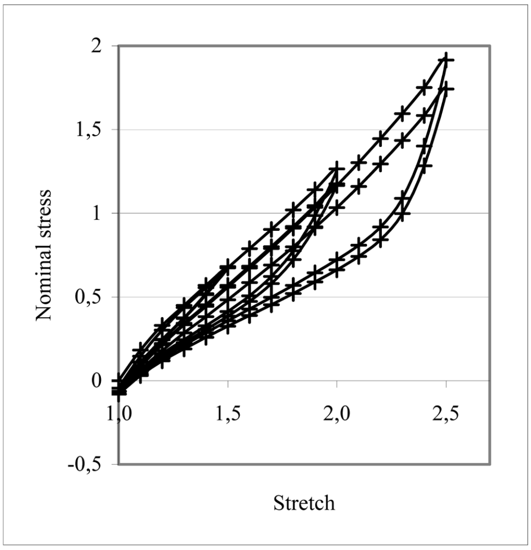


Fig. 5 Numerical simulation of uniaxial tension under cyclic loading with maximum stretches of $\lambda = 1.5$, $\lambda = 2.0$ and $\lambda = 2.5$: nominal stress-stretch curves

A Comparison between numerical simulation and experimental data with 20 phr (by volume) of Carbon Black filler with maximum stretch $\lambda = 3.0$ is shown in Fig. 3, in which the comparison results of primary loading and unloading and the comparison results of reloading and secondary unloading are given in Fig. 3(a) and 3(b), respectively. The numerical results are in good agreement with the experimental data. Fig. 4 illustrates the whole evolution of the cyclic loading process.

Fig. 5 demonstrates the numerical calculation with 20 phr (by volume) of Carbon Black filler with different values of maximum stretch $\lambda = 1.5$, $\lambda = 2.0$ and $\lambda = 2.5$. The values of all parameters used in this simulation were the same as the values listed in Table 1, because the specimens used in the two different experiments were made of the same material. In these numerical simulations, the evolution of stress softening, the permanent deformation and all turning points, on which the loading curve changes from the loading path to the unloading path or from the unloading path to the loading path, were well reproduced compared to experimental data (Fig. 1b).

5. Conclusions

Combination of the pseudo-elastic concept and Gao's model was used to construct a specific model for the description of Mullins effect with permanent deformation. The incorporation of a damage variable and an extra term in the strain-energy function leads to a modified elastic strain energy function to account for different evolutions of primary loading, unloading, reloading or secondary unloading processes. The new damage variable totally involves five model parameters and could be estimated separately according to the different branches of evolution curves. Model parameters are estimated based on experimental data of uniaxial tension with certain maximum stretch under cyclic loading. The numerical results are in good agreement with the experimental data. The values of these parameters are also used to simulate uniaxial tension of a different specimen made of the same material but with different maximum stretches under cyclic loading. In these numerical simulations, the evolution of stress softening, the permanent deformation and all turning points, on which the loading curve changes from the loading path to the unloading path or from the unloading path to the loading path, were well reproduced compared to experimental data.

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