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Reflection of plane waves from the boundary of a thermo-magneto-electroelastic solid half space

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Abstract. The theory of generalized thermo-magneto-electroelasticity is employed to obtain the plane wave solutions in an unbounded, homogeneous and transversely isotropic medium. Reflection phenomena of plane waves is considered at a stress free and thermally insulated surface. For incidence of a plane wave, the expressions of reflection coefficients and energy ratios for reflected waves are derived. To explore the characteristics of reflection coefficients and energy ratios, a quantitative example is set up. The half-space of the thermo-magneto-electroelastic medium is assumed to be made out of lithium niobate. The dependence of reflection coefficients and energy ratios on the angle of incidence is illustrated graphically for different values of electric, magnetic and thermal parameters.

Keywords: thermo-magneto-electroelasticity; plane waves; reflection coefficients; energy ratios

1. Introduction

Ogden and Steigmann (2011) and Dorfmann and Ogden (2014) have presented a monograph on the nonlinear theory of electroelastic and magnetoelastic interactions. Magneto-electro-elastic materials display the coupling behavior among electric, magnetic and mechanical fields. Magneto-electro-elastic materials have various applications due to their ability of converting energy from one kind to another. These materials have been used in lasers, supersonic devices, microwave and infrared applications. The theories of magneto-elasticity, thermo-magneto-elasticity and thermo-magneto-electro-elasticity study the effects of magnetic and electric interactions on an elastic or thermoelastic body. Thermo-magneto-electroelastic materials have been extensively used as electric packaging, sensors and actuators. Wave propagation in thermo-magneto-electroelastic solid has possible applications due to wide use of piezoelectric and piezomagnetic materials in aerospace, automobile and various other industries. The theory of thermo-magneto-electroelasticity has been developed due to the significant contributions of various researchers. For example, Kaliski (1985) has developed the wave equations of thermo-magneto-electroelasticity. Coleman and Dill (1971) have proposed the thermodynamic restrictions on the constitutive equations of electromagnetic theory. Amendola (2000) has determined the restrictions imposed on the assumed constitutive equations by thermodynamics. Li (2003) has presented the uniqueness and reciprocity theorems for thermo-electro-magnet-elasticity without the positive definiteness condition of the elastic parameters. Aouadi (2007) has developed the field equations for Lebon's model of generalized thermo-magneto-electroelasticity.

Various static and dynamic problems in elastic solids with electric, magnetic and thermal effects have been

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studied by many researchers. Some of notable and related contributions are mentioned herein. Paria (1962) has studied the propagation of plane waves in a thermoelastic medium subjected to the magnetic field and has shown that the magnetic effect on plane waves is insignificant for large electrical conductivity. Nayfeh and Nemat-Nasser (1971, 1972) have studied the plane harmonic waves in unbounded thermoelastic and electro-magnetic-thermoelastic media. Roychoudhuri and Chatterjee (1990) have investigated the thermal-shock induced magneto-thermoelastic wave in a perfectly conducting elastic half-space. Hsieh (1990) has presented a detailed review of the mechanical behaviour of new electromagnetic materials and their applications. Ezzat (1997) has introduced the state space formulation in a perfectly conducting medium using the generalized magneto-thermoelasticity. Sherief and Youssef (2004) have investigated the wave propagation in an electro-magneto-thermoelastic half-space whose surface is subjected to thermal shock. Baksi and Bera (2005) have studied the disturbances in an electrically conducting infinite orthotropic thermoelastic elastic solid pervaded by a primary magnetic field with instantaneous point heat source. Das and Kanoria (2009) have employed the generalized thermoelasticity with energy dissipation and have studied the time-harmonic plane wave propagation in a perfectly electrically conducting elastic medium under primary uniform magnetic field. Dai and Rao (2011) have explored the electro-magneto-thermo-elastic behaviors of a hollow sphere composed of functionally graded piezoelectric material under electric, thermal and mechanical loads. Ponnusamy and Selvamani (2012) have discussed the dispersion analysis of magneto-thermoelastic waves in a transversely isotropic cylindrical panel. Abo-Dahab and Singh (2013) have studied the effects of rotation, magnetic field, voids and initial stress on the reflection phenomena in context of the thermoelasticity without energy dissipation. Zhang (2013) has discussed the reflection and refraction phenomena at an interface between transversely isotropic magneto-electro-elastic and non-viscous liquid half-spaces. Kondaiah *et al*. (2013) have studied the pyroelectric and pyromagnetic effects on behavior of magneto-electro-elastic plate under different boundary conditions. Zhang *et al*. (2014) have investigated the propagation of Rayleigh waves in a magneto-electro-elastic half-space with initial stress. Abd-Alla and Othman (2016) have studied the effect of magnetic field in vacuum on the reflection of plane harmonic waves from a semi-infinite thermoelastic solid. Tiwari and Mukhopadhyay (2017) have studied the propagation of electro-magneto-thermoelastic plane waves in the context Green and Naghdi type-II theory of thermoelasticity. Vinyas and Kattimani (2017) have analyzed the multiphysics response of magneto-electro-elastic cantilever beam under thermo-mechanical loadings. Yakhno (2018) has suggested a new method to find an explicit solution of an initial value problem for governing equations of magneto-electro-elasticity. Moreno-Navarro *et al*. (2018) have proposed a fully-coupled thermodynamic-based transient finite element formulation for interactions of electric, magnetic, thermal and mechanic fields in a linear case. Lata and Kaur (2019) have studied the effects of two-temperature and rotation on thermo-mechanical interactions in a transversely isotropic magneto-thermoelastic solids.

Sarkar *et al*. (2019) have studied the reflection phenomenon of the magneto-thermoelastic plane waves from a stress-free surface of a homogeneous, isotropic, thermally and electrically conducting solid half-space in context of the generalized thermoelasticity model with memory-dependent derivative. Sarkar and De (2020) have employed the modified Green-Lindsay model of generalized thermoelasticity to study the propagation of time-harmonic plane waves in an infinite elastic solid. Singh (2020) have studied the plane wave characteristics in context of two-temperature porothermoelasticity. Singh *et al*. (2016) have used the theories by Aouadi (2007), Lord and Shulman (1967) and Dhaliwal and Sherief (1980) to develop the governing equations of generalized thermo-magneto-electroelasticity which incorporate a flux-rate term into Fourier's law of heat conduction. They have shown that there exist three plane waves, namely, quasi-P (qP), quasi-SV (qSV) and quasi-T (qT) waves.

To the knowledge of authors, no work has been reported till date on the reflection phenomena in context of the theories developed by Aouadi (2007), Lord and Shulman (1967) and Dhaliwal and Sherief (1980). The objective of this paper is to study reflection of the plane waves from a stress free and thermally insulated surface of a generalized thermo-magneto-electro-elastic solid half space. The expressions of reflection coefficients and energy ratios are obtained analytically. Using material parameters of lithium niobate ($LiNbO_3$), the reflection coefficients and energy ratios of reflected waves are computed and illustrated graphically against the angle of incidence for different material parameters.

2. Basic equations

Following Aouadi (2007), Lord and Shulman (1967) and Dhaliwal and Sherief (1980), the field equations governing the theroy of generalized thermo-magneto-electro-elasticity are formulated as

The equations of motion

$$\sigma_{ji,j} + F_i = \rho \ddot{u}_i, \tag{1}$$

The equations of the electric and magnetic fields

$$D_{i,i} = \rho_0, \;\; B_{i,i} = \sigma, \tag{2}$$

The energy equation

$$\rho T_0 \dot{\eta} = q_{i,i} + \rho h, \tag{3}$$

The constitutive equations

$$\sigma_{ij} = c_{ijkl} e_{kl} + F_{ijk}\zeta_k + \lambda_{ijk} E_k - a_{ij} T, \tag{4}$$

$$D_k = -\lambda_{kij} e_{ij} + \alpha_{ki}\zeta_i + \gamma_{ki} E_i + p_k T, \tag{5}$$

$$B_k = -F_{kij} e_{ij} + A_{ki}\zeta_i + \alpha_{ki} E_i + m_k T, \tag{6}$$

$$\rho\eta = a_{ij} e_{ij} + m_k \zeta_k + p_k E_k + c_e T, \tag{7}$$

$$k_{ij} T_{,j} = q_i + \tau_0 \dot{q}_i, \tag{8}$$

and the geometrical equations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad E_i = -\psi_{,i}, \quad \zeta_i = -\phi_{,i}, \tag{9}$$

where symbols have their usual meanings. Here, the subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate. The superposed dot denotes the partial differentiation with respect to time t. The following symmetries hold between the constitutive parameters

$$c_{ijkl} = c_{klij} = c_{jikl}, \qquad \lambda_{ijk} = \lambda_{kij} = \lambda_{kji}, \qquad F_{ijk} = F_{kij} = F_{kji},$$
$$a_{ij} = a_{ji}, \; \gamma_{ij} = \gamma_{ji}, \; \alpha_{ij} = \alpha_{ji}. \tag{10}$$

3. Two-dimensional specialization

We consider an infinite, homogeneous and transversely isotropic thermo-magneto-electroelastic medium at uniform temperature T_0, initial electric potential ψ_0 and initial magnetic potential ϕ_0. The medium is taken transversely isotropic in such a way that the planes of isotropy become perpendicular to the z-axis. The origin is taken at any point on the plane surface $z = 0$ and the x-axis is taken along the propagation direction. For a plane strain parallel to $x - z$ plane with displacement vector $\vec{u} = (u_1, 0, u_3)$, electric potential $\psi(x,z,t)$, magnetic potential $\phi(x,z,t)$ and temperature $T(x,z,t)$, the equations of a transversely isotropic thermo-magneto-electroelastic medium in $x - z$ plane are formulated after rejecting the dependency of y-direction as well as the derivative with respect to y. With the help of the symmetry conditions (10) and Eqs. (4) to (9), the Eqs. (1) to (3) reduce to the following system of five partial differential equations in u_1, u_3, ϕ, ψ and T in absence of body forces, electric charge density, electric current density and heat supply

$$c_{11}\frac{\partial^2 u_1}{\partial x^2} + c_{31}\frac{\partial^2 u_3}{\partial x \partial z} - F_{11}\frac{\partial^2 \phi}{\partial x^2} - 2F_{31}\frac{\partial^2 \phi}{\partial x \partial z} - \lambda_{11}\frac{\partial^2 \psi}{\partial x^2}$$
$$-2\lambda_{31}\frac{\partial^2 \psi}{\partial x \partial z} - a_1 \frac{\partial T}{\partial x} + c_{55}\left(\frac{\partial^2 u_1}{\partial z^2} + \frac{\partial^2 u_3}{\partial x \partial z}\right) = \rho\frac{\partial^2 u_1}{\partial t^2}, \tag{11}$$

$$c_{55}\left(\frac{\partial^2 u_3}{\partial x^2}+\frac{\partial^2 u_1}{\partial x\partial z}\right)-F_{31}\frac{\partial^2\phi}{\partial x^2}-\lambda_{31}\frac{\partial^2\psi}{\partial x^2}+c_{31}\frac{\partial^2 u_1}{\partial x\partial z}$$
$$+c_{33}\frac{\partial^2 u_3}{\partial z^2}-F_{33}\frac{\partial^2\phi}{\partial z^2}-\lambda_{33}\frac{\partial^2\psi}{\partial z^2}-a_3\frac{\partial T}{\partial z}=\rho\frac{\partial^2 u_3}{\partial t^2}, \tag{12}$$

$$\lambda_{11}\frac{\partial^2 u_1}{\partial x^2}+\lambda_{31}\left(2\frac{\partial^2 u_1}{\partial x\partial z}+\frac{\partial^2 u_3}{\partial x^2}\right)+\lambda_{33}\frac{\partial^2 u_3}{\partial z^2}+\alpha_1\frac{\partial^2\phi}{\partial x^2}$$
$$+\alpha_3\frac{\partial^2\phi}{\partial z^2}+\gamma_1\frac{\partial^2\psi}{\partial x^2}+\gamma_3\frac{\partial^2\psi}{\partial z^2}-p_1\frac{\partial T}{\partial x}-p_3\frac{\partial T}{\partial z}=0, \tag{13}$$

$$F_{11}\frac{\partial^2 u_1}{\partial x^2}+F_{31}\left(2\frac{\partial^2 u_1}{\partial x\partial z}+\frac{\partial^2 u_3}{\partial x^2}\right)+F_{33}\frac{\partial^2 u_3}{\partial z^2}+A_1\frac{\partial^2\phi}{\partial x^2}$$
$$+A_3\frac{\partial^2\phi}{\partial z^2}+\alpha_1\frac{\partial^2\psi}{\partial x^2}+\alpha_3\frac{\partial^2\psi}{\partial z^2}-m_1\frac{\partial T}{\partial x}-m_3\frac{\partial T}{\partial z}=0, \tag{14}$$

$$\left(1+\tau_0\frac{\partial}{\partial t}\right)T_0[a_1\frac{\partial^2 u_1}{\partial x\partial t}+a_3\frac{\partial^2 u_3}{\partial z\partial t}-m_1\frac{\partial^2\phi}{\partial x\partial t}-m_3\frac{\partial^2\phi}{\partial z\partial t}-p_1\frac{\partial^2\psi}{\partial x\partial t}$$
$$-p_3\frac{\partial^2\psi}{\partial z\partial t}+c_e\frac{\partial T}{\partial t}]=K_1\frac{\partial^2 T}{\partial x^2}+K_3\frac{\partial^2 T}{\partial z^2}. \tag{15}$$

where

$$c_{11}=c_{1111},\quad c_{33}=c_{3333},\ c_{31}=c_{3311}=c_{1133},\quad c_{55}=c_{3113},\quad F_{11}=F_{111},$$
$$F_{31}=F_{311},\ \lambda_{11}=\lambda_{111},\quad \lambda_{31}=\lambda_{311},\quad a_1=a_{11},\quad a_3=a_{33},\quad \alpha_1=\alpha_{11},$$
$$\alpha_3=\alpha_{33},\ \gamma_1=\gamma_{11},\ \gamma_3=\gamma_{33}, K_1=k_{11},\ K_3=k_{33},\ A_1=A_{11},\ A_3=A_{33}.$$

4. Plane wave propagation

The plane harmonic solutions of Eqs. (11) to (15) are sought in the following form

$$\{u_1,\ u_3,\ T,\ \psi,\ \phi\}=\{\bar{u}_1,\ \bar{u}_3,\ \bar{T},\ \bar{\psi},\ \bar{\phi}\}e^{\iota k(\sin\theta\, x+\cos\theta\, z-vt)}, \tag{16}$$

where $\iota=\sqrt{-1}$, θ is the angle of propagation, k is the wave number, v is the complex wave speed and $\bar{u}_1,\ \bar{u}_3,\ \bar{T},\ \bar{\psi},\ \bar{\phi}$ are arbitrary constants.

Using Eq. (16) in Eqs. (11) to (15), we obtain the following homogeneous system of five equations in $\bar{u}_1, \bar{u}_3, \bar{T}, \bar{\psi}$ and $\bar{\phi}$.

$$(\zeta-D_1)\bar{u}_1-L_1\bar{u}_3+(\iota/k)a_1\sin\theta\bar{T}+L_2\bar{\psi}+L_3\bar{\phi}=0, \tag{17}$$

$$-L_1\bar{u}_1+(\zeta-D_2)\bar{u}_3+(\iota/k)a_3\cos\theta\bar{T}+D_3\bar{\psi}+D_6\bar{\phi}=0, \tag{18}$$

$$\bar{a}_1T_0\zeta\sin\theta\bar{u}_1+\bar{a}_3T_0\zeta\cos\theta\bar{u}_3-(\iota/k)(D_8-\bar{c}_e\zeta)\bar{T}-\zeta\bar{P}\bar{\psi}-\zeta\bar{M}\bar{\phi}=0, \tag{19}$$

$$L_2\bar{u}_1+D_3\bar{u}_3+(\iota/k)P\bar{T}+D_4\bar{\psi}+D_5\bar{\phi}=0, \tag{20}$$

$$L_3\bar{u}_1+D_6\bar{u}_3+(\iota/k)M\bar{T}+D_5\bar{\psi}+D_7\bar{\phi}=0, \tag{21}$$

where $\zeta=\rho v^2$ and the expressions for D_i (i = 1, 2,..,8), L_j (j = 1, 2, 3), M and P are given in Appendix I. The system has non-trivial solution if the determinant of the coefficients of $\bar{u}_1,\ \bar{u}_3,\ \bar{T},\ \bar{\psi},\ \bar{\phi}$ vanishes, i.e.,

$$A\zeta^3+B\zeta^2+C\zeta+D=0, \tag{22}$$

where the expressions for A, B, C and D are given in Appendix I. The dispersion Eq. (22) is a cubic equation in v^2 with complex coefficients. Therefore, the three roots of Eq. (22) are complex. The square of complex phase velocities $v_j{}^2, (j=1,2,3)$ of quasi-waves will vary with the direction of phase propagation. Then, the complex phase velocities of the quasi-waves ($v_j=r_j+\iota s_j$) defines the phase propagation velocities

$V_j = (r_j^2 + s_j^2)/r_j$ and attenuation quality factors $q_j = -2s_j/r_j$ for each j. Therefore, three propagating waves in thermo-magneto-electroelastic medium are attenuating waves. For real slowness vector, the directions of propagation and attenuation are same. Hence, these waves are homogeneous waves. The three roots $v_j{}^2, (j = 1,2,3)$ of Eq. (22) correspond to quasi-P (qP), quasi-T (qT) and quasi-SV (qSV) waves, respectively. If we neglect electric, magnetic and thermal fields, the Eq. (22) reduces to

$$\zeta^2 + (D_1 + D_2)\zeta + (D_1 D_2 - L_1{}^2) = 0, \tag{23}$$

which gives the speeds of qausi-P and quasi-SV waves in a transversely isotropic elastic media.

5. Reflection from the stress free surface

In this section, the reflection of qP wave from a stress free, charge free and thermally insulated surface $z = 0$ is studied. For an incident qP wave propagating through *x*-*z* plane, the qP, qT and qSV waves get reflected back into the half-space. The complete geometry showing the half-space with the incident and reflected waves has been shown in Fig. 1.

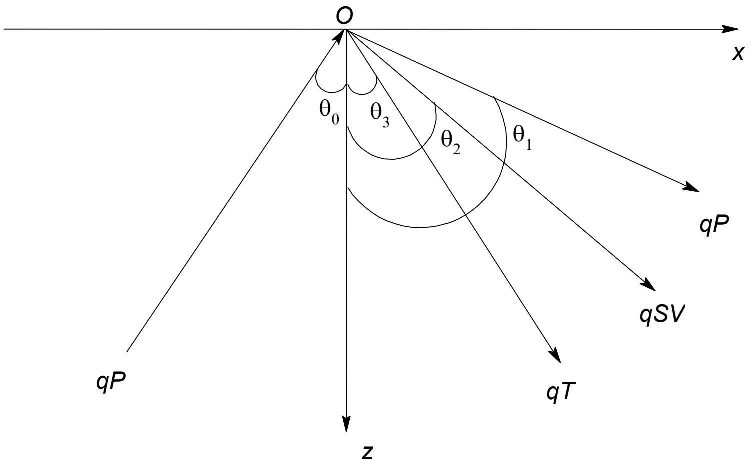


Fig. 1 Geometry of the half-space showing incident and reflected waves

The relevant boundary conditions at stress free and thermally insulated surface $z = 0$ are taken as

$$\sigma_{33} = 0,\ \sigma_{31} = 0,\ \frac{\partial T}{\partial z} = 0, \tag{24}$$

where

$$\sigma_{33} = c_{31}u_{1,1} + c_{33}u_{3,3} - F_{33}\phi_{,3} - \lambda_{33}\psi_{,3} - a_3 T,$$
$$\sigma_{31} = c_{55}(u_{1,3} + u_{3,1}) - F_{31}\phi_{,1} - \lambda_{31}\psi_{,1}.$$

The appropriate displacement components, temperature, electric and magnetic potentials of incident and reflected waves in the half-space $z \geq 0$ are

$$u_1 = A_0 e^{\iota k_1(\sin\theta_0 x + \cos\theta_0 z - V_1 t)} + A_1 e^{\iota k_1(\sin\theta_1 x - \cos\theta_1 z - V_1 t)} + A_2 e^{\iota k_2(\sin\theta_2 x - \cos\theta_2 z - V_2 t)} + A_3 e^{\iota k_3(\sin\theta_3 x - \cos\theta_3 z - V_3 t)}, \tag{25}$$

$$u_3 = \eta_0 A_0 e^{\iota k_1(\sin\theta_0 x + \cos\theta_0 z - V_1 t)} + \eta_1 A_1 e^{\iota k_1(\sin\theta_1 x - \cos\theta_1 z - V_1 t)} + \eta_2 A_2 e^{\iota k_2(\sin\theta_2 x - \cos\theta_2 z - V_2 t)} + \eta_3 A_3 e^{\iota k_3(\sin\theta_3 x - \cos\theta_3 z - V_3 t)}, \tag{26}$$

$$T = \zeta_0 A_0 e^{\iota k_1(\sin\theta_0 x + \cos\theta_0 z - V_1 t)} + \zeta_1 A_1 e^{\iota k_1(\sin\theta_1 x - \cos\theta_1 z - V_1 t)} + \zeta_2 A_2 e^{\iota k_2(\sin\theta_2 x - \cos\theta_2 z - V_2 t)} + \zeta_3 A_3 e^{\iota k_3(\sin\theta_3 x - \cos\theta_3 z - V_3 t)}, \tag{27}$$

$$\psi = \xi_0 A_0 e^{\iota k_1(\sin\theta_0 x + \cos\theta_0 z - V_1 t)} + \xi_1 A_1 e^{\iota k_1(\sin\theta_1 x - \cos\theta_1 z - V_1 t)} + \xi_2 A_2 e^{\iota k_2(\sin\theta_2 x - \cos\theta_2 z - V_2 t)} + \xi_3 A_3 e^{\iota k_3(\sin\theta_3 x - \cos\theta_3 z - V_3 t)}, \tag{28}$$

$$\phi = \chi_0 A_0 e^{\iota k_1(\sin\theta_0 x + \cos\theta_0 z - V_1 t)} + \chi_1 A_1 e^{\iota k_1(\sin\theta_1 x - \cos\theta_1 z - V_1 t)} + \chi_2 A_2 e^{\iota k_2(\sin\theta_2 x - \cos\theta_2 z - V_2 t)} + \chi_3 A_3 e^{\iota k_3(\sin\theta_3 x - \cos\theta_3 z - V_3 t)}, \tag{29}$$

where the coupling coefficients $\eta_i,\ \zeta_i,\ \xi_i,\ \chi_i\ (i = 0, 1, 2, 3)$ are given in Appendix II.

The solutions (25) to (29) for incident and reflected waves satisfy the boundary conditions (24) if following relation similar to the Snell′s law holds

$$\frac{\sin\theta_0}{V_1} = \frac{\sin\theta_i}{V_i},\ (i = 1, 2, 3) \tag{30}$$

With the help of the Snell’s law (30), we obtain the following expressions for reflection coefficients as

$$\frac{A_1}{A_0} = \frac{\Delta_1}{\Delta},\ \frac{A_2}{A_0} = \frac{\Delta_2}{\Delta},\ \frac{A_3}{A_0} = \frac{\Delta_3}{\Delta}, \tag{31}$$

where

$$\Delta = a_{11}(b_{12}d_{13} - b_{13}d_{12}) - a_{12}(b_{11}d_{13} - b_{13}d_{11}) + a_{13}(b_{11}d_{12} - b_{12}d_{11}),$$
$$\Delta_1 = (b_{13} - a_{13})(d_{12} + a_{12}) - (b_{12} - a_{12})(d_{13} + a_{13}),$$
$$\Delta_2 = (b_{11} - a_{11})(d_{13} + a_{13}) - (b_{13} - a_{13})(d_{11} + a_{11}),$$
$$\Delta_3 = (b_{12} - a_{12})(d_{11} + a_{11}) - (b_{11} - a_{11})(d_{12} + a_{12}),$$
$$a_{1j} = \frac{c_{31}\sin\theta_0\left(\frac{V_j}{V_1}\right) + Q_j Q_{1j} + \iota a_3\left(\frac{\zeta_j}{k_j}\right)}{c_{31}\sin\theta_0 + \cos\theta_0 Q_{10} + \iota a_3\left(\frac{\zeta_1}{k_1}\right)}\left(\frac{V_1}{V_j}\right),\quad (j = 1, 2, 3),$$
$$b_{1j} = \frac{-c_{55} Q_j + \sin\theta_0 Q_{2j}\left(\frac{V_j}{V_1}\right)}{c_{55}\cos\theta_0 + \sin\theta_0 Q_{20}}\left(\frac{V_1}{V_j}\right),\quad d_{1j} = \frac{Q_j\left(\frac{\zeta_j}{k_j}\right)}{\cos\theta_0\left(\frac{\zeta_1}{k_1}\right)}\left(\frac{V_1}{V_j}\right)^2,\quad (j = 1, 2, 3),$$

and

$$Q_{10} = (-F_{33}\chi_0 - \lambda_{33}\xi_0 + c_{33}\eta_0),\ Q_{20} = (-F_{31}\chi_0 - \lambda_{31}\xi_0 + c_{55}\eta_0),$$
$$Q_{1j} = (F_{33}\chi_j + \lambda_{33}\xi_j - c_{33}\eta_j),\quad Q_{2j} = (-F_{31}\chi_j - \lambda_{31}\xi_j + c_{55}\eta_j),$$
$$Q_j = \sqrt{1 - \left(\frac{V_j}{V_1}\right)^2 \sin^2\theta_0},\ (j = 1, 2, 3).$$

Following Achenbach (1973), the expression for time average of power per unit area are given as

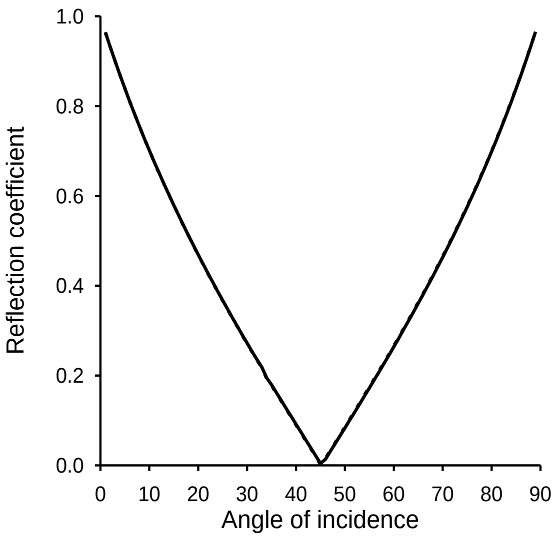
$$< P^* > = \sigma_{33}\dot{u}_3 + \sigma_{31}\dot{u}_1. \tag{32}$$

Using the Eq. (32), the time average power per unit area $< P^* >$ of incident and reflected waves are obtained. Then, the expressions for energy ratios $ER_j\ (j = 1, 2, 3)$ are obtained as

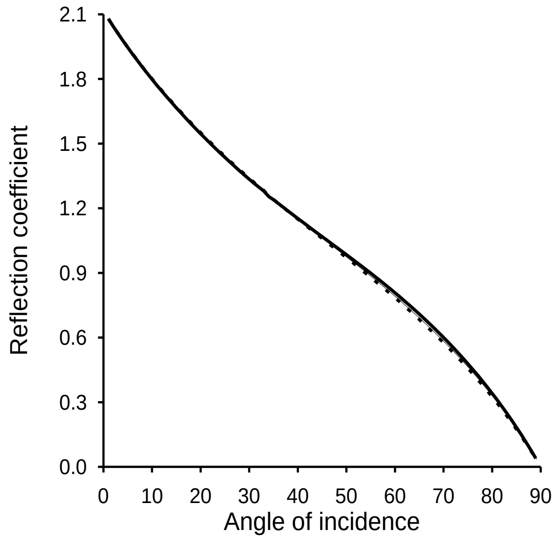
$$ER_j = \frac{X_j\,\eta_j - c_{55} Q_j + \left(\frac{V_j}{V_1}\right)\sin\theta_0 Q_{2j}}{X_0\,\eta_0 + c_{55}\cos\theta_0 + \sin\theta_0 Q_{20}}\left(\frac{V_1}{V_j}\right)\left(\frac{A_j}{A_0}\right)^2, \tag{33}$$

where

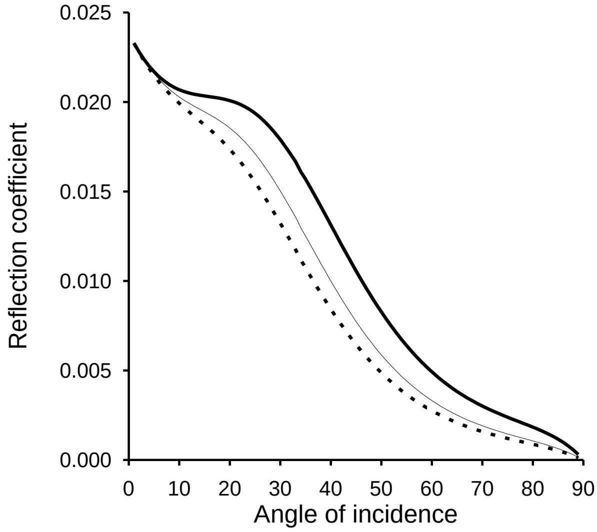
$$X_j = c_{31}\sin\theta_0\left(\frac{V_j}{V_1}\right) + Q_j\, Q_{1j} + \iota a_3\ \left(\frac{\zeta_j}{k_j}\right),\quad X_0 = c_{31}\sin\theta_0 + \cos\theta_0 Q_{10} + \iota a_3\ \left(\frac{\zeta_0}{k_1}\right).$$



(a) Reflected qP wave



(b) Reflected qSV wave



(c) Reflected qT wave

Fig. 2 (a)-(c) Variations of reflection coefficients of reflected waves for incidence of qP wave when K_1=4 (Thick solid curve), K_1=8 (Thin solid curve) and K_1=12 (Dashed curve)

6. Numerical results and discussion

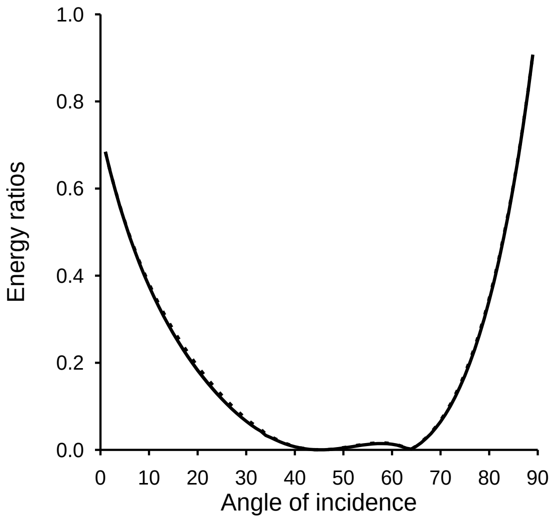
The following physical constants of Lithium Niobate (Weis and Gaylord,\ 1985) have been chosen to compute the reflection coefficients and energy ratios of reflected qP, qT and qSV waves:

$$\rho = 4.647 \times 10^3 Kgm^{-3}, \quad c_{11} = 2.03 \times 10^{11} Nm^{-2}, \quad c_{33} = 2.424 \times 10^{11} Nm^{-2},$$
$$c_{55} = 0.595 \times 10^{11} Nm^{-2}, \quad c_{31} = 0.752 \times 10^{11} Nm^{-2}, \quad \lambda_{11} = 1.13 Cm^{-2},$$
$$\lambda_{33} = 1.33 Cm^{-2}, \quad \lambda_{31} = 0.23 Cm^{-2}, \quad \gamma_1 = 85.2, \quad \gamma_1 = 28.7,$$
$$F_{11} = 0.2 \times 10^{-2} Kg, \quad F_{33} = 0.15 \times 10^{-2} Kg, \quad F_{31} = 0.1 \times 10^{-2} Kg,$$
$$A_1 = 0.005\, NA^{-2}, \quad A_3 = 0.004\, NA^{-2}, \; a_1 = 13.3 \times 10^{-6} K^{-1},$$

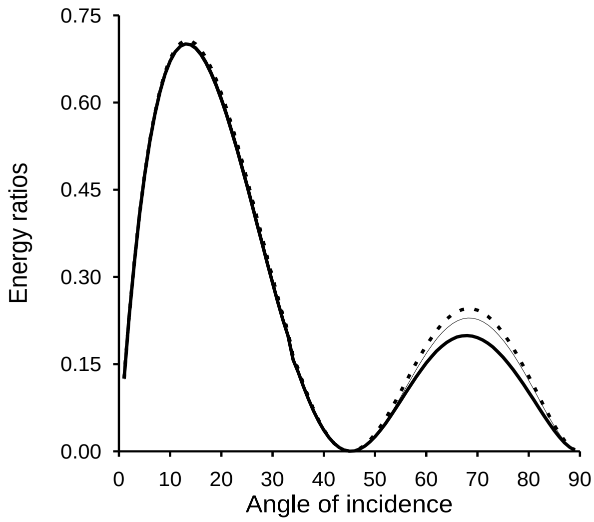
$$a_3 = 10.3 \times 10^{-6} K^{-1}, \quad K_1 = 4 W m^{-1} K^{-1}, \quad K_3 = 4.2 W m^{-1} K^{-1},$$
$$\alpha_1 = 0.02\, C m^{-1} A^{-1}, \quad \alpha_3 = 0.03 C m^{-1} A^{-1}, \quad m_1 = 0.006 N m^{-1} A^{-1} K^{-1},$$
$$m_3 = 0.004 N m^{-1} A^{-1} K^{-1}, \ \tau_0 = 0.00000005 \text{ s},$$
$$p_1 = 0.133 \times 10^5 N C^{-1} K^{-1}, \ p_3 = 0.103 \times 10^5 N C^{-1} K^{-1}.$$

Using the above material parameters in Eqs. (31) and (33), the reflection coefficients and energy ratios of reflected qP, qSV and qT waves are computed numerically for incidence of qP wave. The reflection coefficients and energy ratios of the reflected waves are illustrated graphically against the angle of incidence θ_0 $(0^o - 90^o)$ in Fig. 2(a) to 2(c) and Fig. 3(a) to 3(c), respectively, when $K_1 = 4$ (thick solid), 8 (thin solid) and 12 (dashed curve).

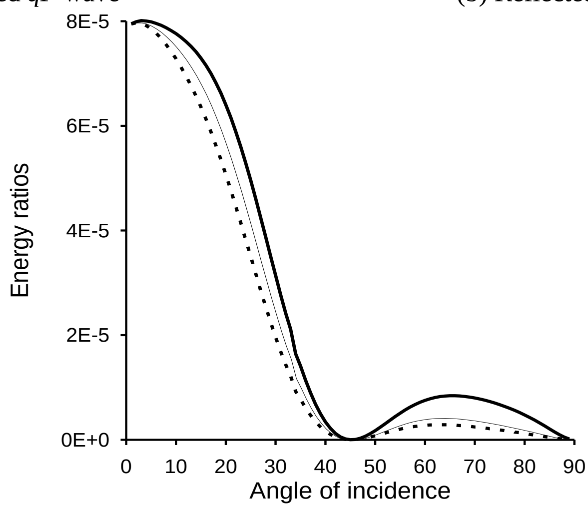
The reflection coefficient of reflected qP wave (as shown in Fig. 2(a)) first decreases very sharply with the angle of incidence and attains a minimum value at $\theta_0 = 45^o$. Thereafter, it increases very sharply to itsd



(a) Reflected qP wave

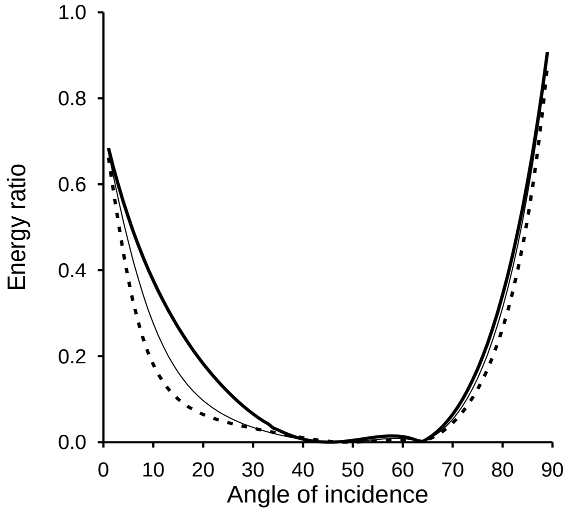


(b) Reflected qSV wave

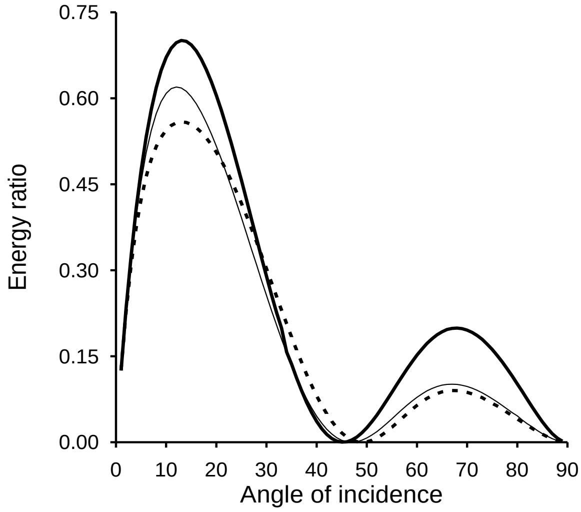


(c) Reflected qT wave

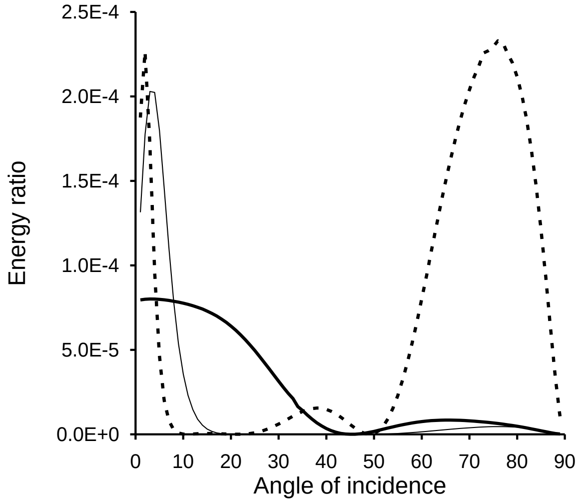
Fig. 3 (a)-(c) Variations of energy ratios of reflected waves for incidence of qP wave when K_1=4 (Thick solid curve), K_1=8 (Thin solid curve) and K_1=12 (Dashed curve)



(a) Reflected qP wave



(b) Reflected qSV wave

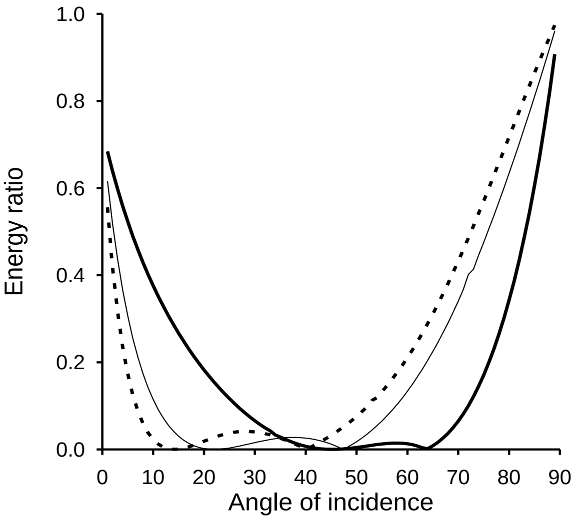


(c) Reflected qT wave

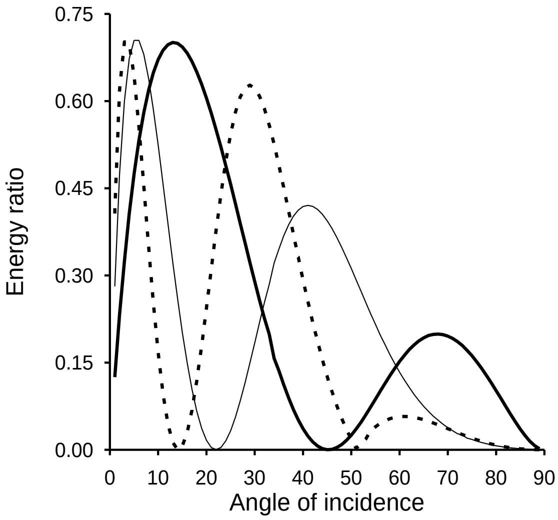
Fig. 4 (a)-(c) Variations of energy ratios of reflected waves for incidence of qP wave when m_1=0.006 (Thick solid curve), m_1=0.1 (Thin solid curve) and m_1=0.2 (Dashed curve)

maximum value at the grazing incidence. The reflection coefficients of reflected qSV and qT waves (as shown in Fig. 2(b) and 2(c)) decreases monotonically as θ_0 varies from 0^o to 90^o. The reflection coefficients of all reflected waves depend on thermal conductivity K_1 at each angle of incidence except normal and grazing incidences. However, the thermal conductivity parameter K_1 affects the reflection coefficients of qT waves more significantly as compared to the reflection coefficients of qP and qSV waves.

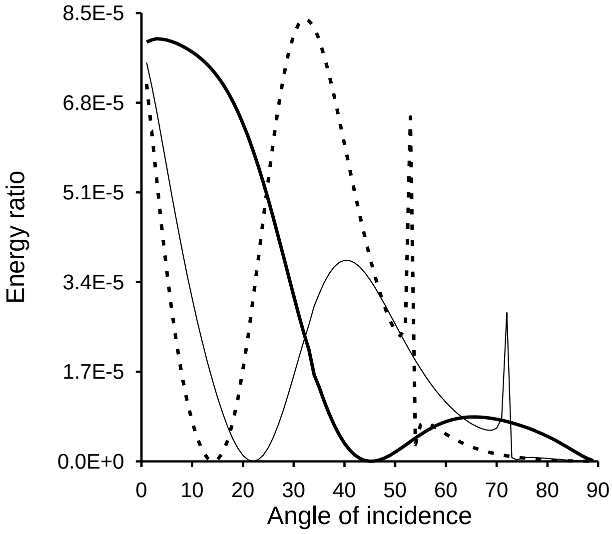
The energy ratios of reflected qP wave (as shown in Fig. 3(a)) for different values of thermal conductivity parameter K_1 monotonically decrease with the angle of incidence and attains their respective minimum value at $\theta_0 = 45^o$. For the range $45^o < \theta_0 \leq 64^o$, these energy ratios oscillates and thereafter these increase very sharply to their respective maxima at grazing incidence. The energy ratios of reflected qSV waves (as shown in Fig. 3(b)) for different values of thermal conductivity parameter K_1 increase very sharply as θ_0



(a) Reflected qP wave



(b) Reflected qSV wave

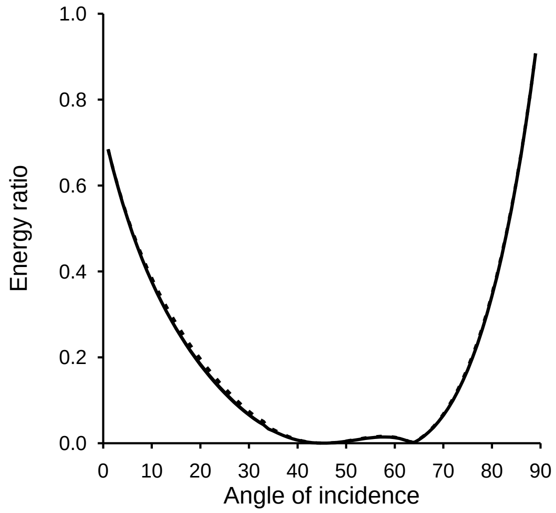


(c) Reflected qT wave

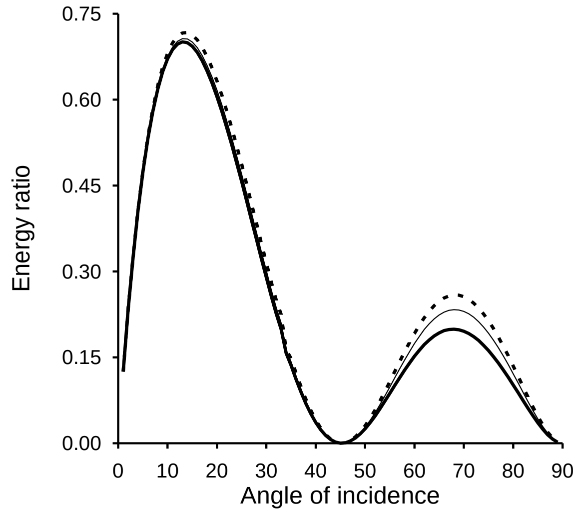
Fig. 5(a)-(c) Variations of energy ratios of reflected waves for incidence of qP wave when F_{11}=0.2×10^{-2} (Thick solid curve), F_{11}=0.5×10^{-2} (Thin solid curve) and F_{11}=0.8×10^{-2} (Dashed curve)

increases and attain respective maxima near $\theta_0 = 10^o$. Beyond $\theta_0 = 10^o$, these energy ratios decreases respective minimum values at $\theta_0 = 45^o$. Thereafter these energy ratios first increase and then decrease till grazing incidence. The energy ratios of reflected qT waves (as shown in Fig. 3(c)) for different values of thermal conductivity parameter K_1 are maximum at normal incidence and these decrease very sharply to resepctive minimum values at $\theta_0 = 45^o$. Thereafter these energy ratios first increase and then decrease till grazing incidence. Similar to the reflection coefficients, the effect of thermal conductivity parameter K_1 is also observed more prominent on the energy ratios of reflected qT as compared to the energy ratios of reflected qP and qSV waves.

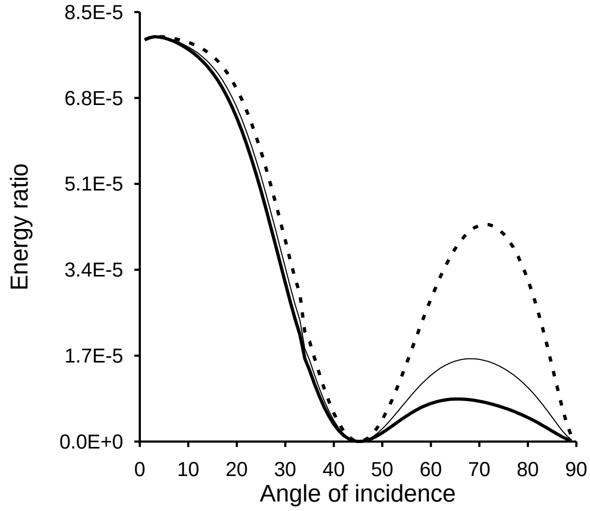
The energy ratios of all reflected waves are also illustrated graphically against the angle of incidence θ_0 in Fig. 4(a) to 4(c) for magneto-thermal parameter $m_1 = 0.006$ (thick solid), 0.1 (thin solid) and 0.2



(a) Reflected qP wave



(b) Reflected qSV wave



(c) Reflected qT wave

Fig. 6(a)-(c) Variations of energy ratios of reflected qP wave for incidence of qP wave when A_1=0.005 (Thick solid curve), A_1=0.01 (Thin solid curve) and A_1=0.02 (Dashed curve)

(dashed curve). The energy ratios of all reflected waves change with magneto-thermal parameter m_1 at each angle of incidence. However, the energy ratio of reflected qT wave changes more significantly due to the change in magneto-thermal parameter m_1.

The energy ratios of reflected waves are also plotted against the angle of incidence θ_0 in Fig. 5(a) to 5(c) for magnetic parameter $F_{11} = 0.2 \times 10^{-2}$ (thick solid), 0.5×10^{-2} (thin solid) and 0.8×10^{-2} (dashed curve). From Fig. 5(a) to 5(c), it is observed that the energy ratios of all reflected waves change significantly due to the change in magnetic parameter F_{11}.

The energy ratios of reflected waves are also illustrated graphically against the angle of incidence θ_0 in Fig. 6(a) to 6(c) for magnetic parameter $A_1 = 0.005$ (thick solid), 0.01 (thin solid) and 0.02 (dashed curve). The magnetic parameter A_1 affects the energy ratios of all reflected waves. But, this effect of magnetic parameter is observed more significant on the energy ratios of reflected qT wave as compared to other reflected waves.

7. Conclusions

The plane harmonic wave solutions of the governing equations of generalized thermo-magneto-electroelasticity are obtained which indicate the possible propagation of three quasi plane waves, namely, qP, qT and qSV waves. For incident qP wave, the reflection coefficients (amplitude ratios) and energy ratios of reflected are analytically obtained. A quantitative example of Lithium Niobate is setup to compute numerically the reflection coefficients and energy ratios of reflection waves. For incident qP wave, the energy share of reflected qP wave is noticed maximum. However, the energy share of reflected qT wave is noticed much smaller as compared to other reflected waves. From theoretical derivations and numerical results, it is found that the amplitude and energy ratios of reflected waves depend upon the frequency, thermal relaxation time, electric, magnetic and thermal parameters. The energy ratios and reflection coefficients of all reflected waves are computed against the angle of incidence for different electric, magnetic and thermal parameters. The different values of thermal, magnetic-thermal and magnetic parameters K_1, m_1, F_{11} and A_1 are chosen for illustrations of energy ratios and reflection coefficients of all reflected waves. It is noticed that all reflected waves are affected due to these parameters. This effect is noticed minimum at normal and grazing incidences. The reflected qT wave is found most affected due to these material parameters. The electric, electric-thermal and electric-magnetic parameters change very slighly the values of the reflection coefficients and energy ratios and hence not illustrated graphically. From numerical computations, it is also noticed that the sum of energy ratios of all reflected waves is found equal or less than unity at each angle of incidence. This fact validates the present numerical results and also justifies the greater than one value of reflection coefficient for reflected qSV at angles near the normal incidence. In absence of electric and magnetic parameters, the present numerical results agree with those published in earlier works (Sharma *et al*. 2003, Das *et al*. 2008, Abd-Alla *et al*. 2016). The present numerical results on plane wave characteristics may have potential applications in developing acoustic/ultrasonic devices, sensors and actuators, electric packaging, magnetic field probes, hydrophones and transducers working in a desired ways.

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Nomenclature

F_i	the body force,
ρ_0	the electric charge density,
σ	the electric current density,
ρ	the mass density,
h	the heat supply,
u_i	the displacement components,
ψ	the electric potential,
ϕ	the magnetic potential,
σ_{ij}	the components of stress tensor,
D_k	the dielectric displacement vector,
B_k	the magnetic intensity,
η	the entropy density,
e_{ij}	the components of strain tensor,
E_i	the electric field,
ζ_i	the magnetic field,
T	the temperature change to a reference temperature T_0,
k_{ij}, K_i	the coeffcients of thermal conductivity,
c_e	the specific heat,
α_{kj}, α_k	the electro-magnetic coefficients,
a_{ij}, a_i	the thermal coefficients,
p_i	the electro-thermal coefficients,
m_i	the magneto-thermal coefficients,
c_{ijkl}, c_{ij}	the elastic coefficients,
γ_{kj}, γ_k, $\lambda_{ijk}, \lambda_{ij}$	the electric coefficients,
$A_{kj,}\, A_k$, F_{ijk}, F_{ij}	the magnetic coefficients,
τ_0	the relaxation time.

Appendix I

The expressions for A, B, C and D are obtained as

$$A = \bar{c}_e(D_4D_7 - D_5^2) + \bar{P}(PD_7 - MD_5) - \bar{M}(PD_5 - MD_4),$$

$$B = D_8(D_4D_7 - D_5^2) - \bar{c}_e[(D_1 + D_2)(D_4D_7 - D_5^2) + 2L_2L_3D_5 - L_2^2D_7 - L_3^2D_4 - D_3^2D_7 + 2D_3D_5D_6 - D_4D_6^2] - \bar{P}[(D_1 + D_2)(PD_7 - MD_5) - a_3\cos\theta(D_3D_7 - D_5D_6) - D_6(PD_6 - MD_3) - a_1\sin\theta(L_2D_7 - L_3D_5) + L_3(ML_2 - PL_3)] + \bar{M}[(D_1 + D_2)(PD_5 - MD_4) + a_3\cos\theta(D_3D_5 - D_4D_6) - D_3(MD_3 - PD_6) + a_1\sin\theta(L_2D_5 - L_3D_4) - L_2(ML_2 - PL_3)] + \bar{a}_3T_0\cos\theta[a_3\cos\theta(-D_4D_7 + D_5^2) + D_3(PD_7 - MD_5) - D_6(PD_5 - MD_4)] + \bar{a}_1T_0\sin\theta[a_1\sin\theta(-D_4D_7 + D_5^2) + L_2(PD_7 - MD_5) - L_3(PD_5 - MD_4)],$$

$$C = D_8[(D_1 + D_2)(D_4D_7 - D_5^2) + L_2^2D_7 - 2L_2L_3D_5 + L_3^2D_4 + D_3^2D_7 - 2D_3D_5D_6 + D_4D_6^2] + \bar{c}_e[(D_1D_2 - L_1^2)(D_4D_7 - D_5^2) - 2L_1L_2D_3D_7 + 2L_1L_3D_3D_5 + 2L_1L_2D_5D_6 - 2L_1L_3D_4D_6 + L_2^2D_2D_7 - 2L_2L_3D_2D_5 + L_2^2D_6^2 - 2L_2L_3D_3D_6 + L_3^2D_2D_4 + L_3^2D_3^2 + D_1D_3^2D_7 - 2D_1D_3D_5D_6 + D_1D_4D_6^2] + \bar{P}[(D_1D_2)(PD_7 - MD_5) + a_3\cos\theta D_1(D_3D_7 - D_5D_6) + D_1D_6(PD_6 - MD_3) - L_1^2(PD_7 - MD_5) - a_3\cos\theta L_1(L_2D_5 - L_3D_4) + L_1D_6(ML_2 - PL_3) - a_1\sin\theta L_1(D_3D_7 - D_5D_6) + a_1\sin\theta D_2(L_2D_7 - L_3D_5) + a_1\sin\theta D_6(L_2D_6 - L_3D_3) + L_1L_3(MD_3 - PD_6) - D_2L_3(ML_2 - PL_3) - a_3\cos\theta L_3(L_2D_6 - L_3D_3)] - \bar{M}[(D_1D_2)(PD_5 - MD_4) + a_3\cos\theta D_1(D_3D_5 - D_4D_6) - D_1D_3(MD_3 - PD_6) - L_1^2(PD_5 - MD_4) - a_3\cos\theta L_1(L_2D_5 - L_3D_4) + L_1D_3(ML_2 - PL_3) - a_1\sin\theta L_1(D_3D_5 - D_4D_6) + a_1\sin\theta D_2(L_2D_5 - L_3D_4) + a_1\sin\theta D_3(L_2D_6 - L_3D_3) + L_1L_2(MD_3 - PD_6) - D_2L_2(ML_2 - PL_3) - a_3\cos\theta L_2(L_2D_6 - L_3D_3)] + \bar{a}_3T_0\cos\theta[a_3\cos\theta D_1(D_4D_7 - D_5^2) - D_1D_3(PD_7 - MD_5) + D_1D_6(PD_5 - MD_4) - a_1\sin\theta L_1(D_4D_7 - D_5^2) - a_1\sin\theta D_3(L_2D_7 - D_5L_3) + a_1\sin\theta D_6(L_2D_5 - L_3D_4) + L_1L_2)(PD_7 - MD_5) + a_3\cos\theta L_2(L_2D_7 - L_3D_3) - L_2D_6(ML_2 - PL_3) - L_1L_3(PD_5 - MD_4) - a_3\cos\theta L_3(L_2D_5 - L_3D_4) + D_3L_3(ML_2 - PL_3)] + \bar{a}_1T_0\sin\theta[a_3\cos\theta L_1(-D_4D_7 + D_5^2) - L_1D_3(PD_7 - MD_5) + L_1D_6(PD_5 - MD_4) + a_1\sin\theta D_2(D_4D_7 - D_5^2) + a_1\sin\theta D_3(D_3D_7 - D_5D_6) - a_1\sin\theta D_6(D_3D_5 - D_4D_6) - L_2D_2(PD_7 - MD_5) - a_3\cos\theta L_2(D_3D_7 - D_5D_6) + L_2D_6(MD_3 - PD_6) - L_3D_2(PD_5 - MD_4) + a_3\cos\theta L_3(D_3D_5 - D_4D_6) - L_3D_3(MD_3 - PD_6)],$$

$$D = -(D_8D_1D_2 + L_1^2)(D_4D_7 - D_5^2) - 2L_1L_2D_3D_7 + 2L_1L_3D_3D_5 - 2L_1L_3D_4D_6 + 2L_1L_2D_5D_6 + L_2^2D_2D_7 - 2L_2L_3D_2D_5 + L_2^2D_6^2 - 2L_2L_3D_3D_6 + L_3^2D_2D_4 + L_3^2D_3^2 + D_1D_3^2D_7 - 2D_1D_3D_5D_6 + D_1D_4D_6^2,$$

and

$$L_1 = (c_{31} + c_{55})\sin\theta\cos\theta, \quad L_2 = 2\lambda_{31}\sin\theta\cos\theta + \lambda_{11}\sin^2\theta,$$
$$L_3 = 2F_{31}\sin\theta\cos\theta + F_{11}\sin^2\theta,$$
$$D_1 = c_{11}\sin^2\theta + c_{55}\cos^2\theta, \quad D_2 = c_{55}\sin^2\theta + c_{33}\cos^2\theta,$$
$$D_3 = \lambda_{31}\sin^2\theta + \lambda_{33}\cos^2\theta, \quad D_4 = \gamma_1\sin^2\theta + \gamma_3\cos^2\theta,$$
$$D_5 = \alpha_1\sin^2\theta + \alpha_3\cos^2\theta, \quad D_6 = F_{31}\sin^2\theta + F_{33}\cos^2\theta,$$
$$D_7 = A_1\sin^2\theta + A_3\cos^2\theta, \quad D_8 = -(K_1\sin^2\theta + K_3\cos^2\theta)/(\tau_0 + \frac{\iota}{\omega}),$$
$$P = p_1\sin\theta + p_3\cos\theta, \quad M = m_1\sin\theta + m_3\cos\theta, \quad \bar{P} = \frac{P}{\rho}, \quad \bar{M} = \frac{M}{\rho},$$
$$\bar{a}_3 = \frac{a_3}{\rho}, \quad \bar{a}_1 = \frac{a_1}{\rho}, \quad \bar{c}_e = \frac{c_e}{\rho}.$$

Appendix II

Making use of the solutions (25) to (29) for incident and reflected waves into the equations (11) to (14), the expressions η_i, χ_i, ξ_i, ζ_i, $(i = 0, 1, 2, 3)$ obtained after using Snell's law (30) as

$$\eta_i = \frac{\Delta_{i1}}{\Delta_i}, \quad \chi_i = \frac{\Delta_{i2}}{\Delta_i}, \quad \xi_i = \frac{\Delta_{i3}}{\Delta_i}, \quad \frac{\zeta_i}{k_i} = \frac{\Delta_{i4}}{\Delta_i}, \tag{34}$$

where

$$\Delta_i = g_{1i}[g_{6i}(g_{11i}g_{16i} - g_{12i}g_{15i}) - g_{7i}(g_{10i}g_{16i} - g_{12i}g_{14i}) + g_{8i}(g_{10i}g_{15i} - g_{11i}g_{14i})] - g_{2i}[g_{5i}(g_{11i}g_{16i} - g_{12i}g_{15i}) - g_{7i}(g_{9i}g_{16i} - g_{12i}g_{13i}) + g_{8i}(g_{9i}g_{15i} - g_{11i}g_{13i})] + g_{3i}[g_{5i}(g_{10i}g_{16i} - g_{12i}g_{14i}) - g_{6i}(g_{9i}g_{16i} - g_{12i}g_{13i}) + g_{8i}(g_{9i}g_{14i} - g_{10i}g_{13i})] -$$

$$g_{4i}[g_{5i}(g_{10i}g_{15i} - g_{11i}g_{14i}) - g_{6i}(g_{9i}g_{15i} - g_{11i}g_{13i}) + g_{7i}(g_{9i}g_{14i} - g_{10i}g_{13i})],$$

$$\Delta_{i1} = (g_{6i} - g_{2i})[(g_{11i} - g_{3i})(g_{16i} - g_{4i}) - (g_{12i} - g_{4i})(g_{15i} - g_{3i})] - (g_{7i} - g_{3i})[(g_{10i} - g_{2i})(g_{16i} - g_{4i}) - (g_{12i} - g_{4i})(g_{14i} - g_{2i})] + (g_{8i} - g_{4i})[(g_{10i} - g_{2i})(g_{15i} - g_{3i}) - (g_{11i} - g_{3i})(g_{14i} - g_{2i})],$$

$$\Delta_{i2} = (g_{7i} - g_{3i})[(g_{9i} - g_{1i})(g_{16i} - g_{4i}) - (g_{12i} - g_{4i})(g_{13i} - g_{1i})] - (g_{5i} - g_{1i})[(g_{11i} - g_{3i})(g_{16i} - g_{4i}) - (g_{12i} - g_{4i})(g_{15i} - g_{3i})] - (g_{8i} - g_{4i})[(g_{9i} - g_{1i})(g_{15i} - g_{3i}) - (g_{11i} - g_{3i})(g_{13i} - g_{1i})],$$

$$\Delta_{i3} = (g_{5i} - g_{1i})[(g_{10i} - g_{2i})(g_{16i} - g_{4i}) - (g_{12i} - g_{4i})(g_{14i} - g_{2i})] - (g_{6i} - g_{2i})[(g_{9i} - g_{1i})(g_{16i} - g_{4i}) - (g_{12i} - g_{4i})(g_{13i} - g_{1i})] + (g_{8i} - g_{4i})[(g_{9i} - g_{1i})(g_{14i} - g_{2i}) - (g_{10i} - g_{2i})(g_{13i} - g_{1i})],$$

$$\Delta_{i4} = (g_{6i} - g_{2i})[(g_{9i} - g_{1i})(g_{15i} - g_{3i}) - (g_{11i} - g_{3i})(g_{13i} - g_{1i})] - (g_{5i} - g_{1i})[(g_{10i} - g_{2i})(g_{15i} - g_{3i}) - (g_{11i} - g_{3i})(g_{14i} - g_{2i})] - (g_{7i} - g_{3i})[(g_{9i} - g_{1i})(g_{14i} - g_{2i}) - (g_{10i} - g_{2i})(g_{13i} - g_{1i})],$$

and

$$g_{1i} = \frac{(c_{31} + c_{55})\left(\frac{V_i}{V_1}\right)\sin\theta_0\sqrt{1 - \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}}{c_{55} + (c_{11} - c_{55})\left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0 - \rho V_i^2},$$

$$g_{2i} = \frac{F_{11}\left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0 - 2F_{31}\left(\frac{V_i}{V_1}\right)\sin\theta_0\sqrt{1 - \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}}{c_{55} + (c_{11} - c_{55})\left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0 - \rho V_i^2},$$

$$g_{3i} = \frac{\lambda_{11}\left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0 - 2\lambda_{31}\left(\frac{V_i}{V_1}\right)\sin\theta_0\sqrt{1 - \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}}{c_{55} + (c_{11} - c_{55})\left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0 - \rho V_i^2},$$

$$g_{4i} = \frac{-\iota a_1\left(\frac{V_i}{V_1}\right)\sin\theta_0}{c_{55} + (c_{11} - c_{55})\left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0 - \rho V_i^2},$$

$$g_{5i} = \frac{-c_{33} + (c_{33} - c_{55})\left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0 + \rho V_i^2}{-(c_{55} + c_{31})\left(\frac{V_i}{V_1}\right)\sin\theta_0\sqrt{1 - \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}},$$

$$g_{6i} = \frac{F_{33} + (F_{31} - F_{33})\left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}{-(c_{55} + c_{31})\left(\frac{V_i}{V_1}\right)\sin\theta_0\sqrt{1 - \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}},$$

$$g_{7i} = \frac{\lambda_{33} + (\lambda_{31} - \lambda_{33})\left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}{-(c_{55} + c_{31})\left(\frac{V_i}{V_1}\right)\sin\theta_0\sqrt{1 - \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}},$$

$$g_{8i} = \frac{\iota a_3 \sqrt{1 - \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}}{-(c_{55} + c_{31}) \left(\frac{V_i}{V_1}\right) \sin\theta_0 \sqrt{1 - \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}},$$

$$g_{9i} = \frac{\lambda_{33} + (\lambda_{31} - \lambda_{33}) \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}{2\lambda_{31} \left(\frac{V_i}{V_1}\right) \sin\theta_0 \sqrt{1 - \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0} - \lambda_{11} \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0},$$

$$g_{10i} = \frac{\alpha_3 + (\alpha_1 - \alpha_3) \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}{2\lambda_{31} \left(\frac{V_i}{V_1}\right) \sin\theta_0 \sqrt{1 - \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0} - \lambda_{11} \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0},$$

$$g_{11i} = \frac{\gamma_3 + (\gamma_1 - \gamma_3) \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}{2\lambda_{31} \left(\frac{V_i}{V_1}\right) \sin\theta_0 \sqrt{1 - \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0} - \lambda_{11} \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0},$$

$$g_{12i} = \frac{\iota p_1 \left(\frac{V_i}{V_1}\right) \sin\theta_0 - \iota p_3 \sqrt{1 - \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}}{2\lambda_{31} \left(\frac{V_i}{V_1}\right) \sin\theta_0 \sqrt{1 - \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0} - \lambda_{11} \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0},$$

$$g_{13i} = \frac{F_{33} + (F_{31} - F_{33}) \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}{2F_{31} \left(\frac{V_i}{V_1}\right) \sin\theta_0 \sqrt{1 - \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0} - F_{11} \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0},$$

$$g_{14i} = \frac{A_3 + (A_1 - A_3) \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}{2F_{31} \left(\frac{V_i}{V_1}\right) \sin\theta_0 \sqrt{1 - \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0} - F_{11} \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0},$$

$$g_{15i} = \frac{\alpha_3 + (\alpha_1 - \alpha_3) \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}{2F_{31} \left(\frac{V_i}{V_1}\right) \sin\theta_0 \sqrt{1 - \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0} - F_{11} \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0},$$

$$g_{16i} = \frac{\iota m_1 \left(\frac{V_i}{V_1}\right) \sin\theta_0 - \iota m_3 \sqrt{1 - \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}}{2F_{31} \left(\frac{V_i}{V_1}\right) \sin\theta_0 \sqrt{1 - \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0} - F_{11} \left(\frac{V_i}{V_1}\right)^2 \sin^2\theta_0}.$$