Advances in Robotics Research, *Vol. 1, No. 3 (2014) 185-198*
DOI: http://dx.doi.org/10.12989/arr.2014.1.3.185

Kinematics of a decoupled series-parallel manipulator by means of screw theory

J. Gallardo-Alvarado*, R. Rodríguez-Castro[a] and A. Sánchez-Rodríguez[b]

Department of Mechanical Engineering, Instituto Tecnológico de Celaya, México

(Received December 14, 2014, Revised December 14, 2014, Accepted February 12, 2015)

Abstract. This work reports on the kinematic analyses of a non-redundant spatial robot built with a translational manipulator assembled in series connection with a parallel wrist. The first mechanism is a fully-decoupled, fully-isotropic and singularity-free translational manipulator while the second mechanism is a spherical parallel manipulator equipped with rotational actuators. The parallel wrist is free of revolute joints with concurrent axes. Numerical examples are provided in order to show the application of the method.

Keywords: decoupled motions; screw theory; series-parallel manipulator; kinematics

1. Introduction

In the mid-1900s Gough developed a six-legged parallel manipulator with the purpose to study the performance of tires under the action of combined loads simulating aero landing operations (Gough 1957, 1962). Nowadays, the invention of Gough is one of the most investigated mechanisms whose topologies are based on parallel kinematic architectures. In fact, the characteristics of the hexapod have been widely discussed by kinematicians. In particular the forward displacement analysis (FDA) of the general hexapod has been one of the most challenging problems of modern kinematics. It is worth to note the formidable work done in order to formulate and simplify the FDA of the general hexapod, see for instance Wampler 1996, Raghavan 1993, Husty 1996, Innocenti 2001, Rolland 2005, Gan *et al*. 2009. An option to overcome this problem is the development of the so-called series-parallel manipulators. The term series-parallel manipulator was proposed by Zoppi *et al.* 2006 and refers to a hybrid mechanism where two parallel manipulators are assembled in series-connection. It is interesting to note that while the FDA of the general hexapod yields a 40-th univariate polynomial equation, which implies a high computational complexity, all solutions of the same analysis stem from two 8-th univariate polynomial equations for most series-parallel manipulators (Gallardo-Alvarado *et al*. 2008). However, the loss of stiffness is a drawback that must be taken into proper account in order to take advantage of the benefits of series-parallel manipulators. On the other hand, in order to simplify

*Corresponding author, Professor, E-mail: jaime.gallardo@itcelaya.edu.mx
[a]Professor, E-mail: ramon.rodriguez@itcelaya.edu.mx
[b]Professor, E-mail: alvaro.sanchez@itcelaya.edu.mx

http://www.techno-press.org/?journal=arr&subpage=7 ISSN: 2287-4976 (Print), 2287-4984 (Online)

the kinematics and control of robot manipulators, parallel manipulators with decoupled kinematics, or decoupled architecture, is a viable option to do it. In that concern, the development of series-parallel manipulators appears to be an excellent option. Romdhane (1999) introduced a robot built with a translational parallel manipulator assembled in series connection with a spherical parallel manipulator, the decoupled motion is evident in that proposal. Tanev (2000) introduced a series-parallel manipulator where closed-form solutions are available for the Inverse/Forward displacement analysis. Zheng *et al.* (2004) reported on the kinematic analysis of a decoupled series-parallel manipulator obtained by assembling in series connection two 3UPU parallel manipulators with different schemes of actuation. Gallardo-Alvarado *et al.* (2010) introduced a 3UPS+2(3RPS) limited-dof robot with partially decoupled kinematics. Altuzarra *et al.* (2010) proposed a method for the type synthesis of lower mobility robots based on partially decoupled equation sets associated to the displacement analysis. Legnani *et al.* (2012) introduced two isotropic robots with decoupled kinematics in desired configurations. In this work a novel six-degrees-of-freedom series-parallel manipulator with decoupled kinematics is introduced.

The rest of the contribution is organized as follows. In section “Description of the series-parallel manipulator” the proposed decoupled robot is outlined. In section “The translational manipulator” the kinematics of the robot responsible to keep constant the orientation of the end-effector platform as observed from the fixed platform is provided. Afterwards, the kinematics of the spherical parallel manipulator is approached in section “The parallel wrist” by means of the theory of screws, regarding with the infinitesimal kinematics. In order to show the application of the method, in section “Case study”, a numerical example concerned with the forward kinematics of the series-parallel manipulator is provided. Finally, some conclusions are given at the end of the contribution.

2. Description of the series-parallel manipulator

The robot under study is depicted in Fig. 1. It consists of a translational manipulator assembled in series connection with a parallel wrist. The robot is equipped with six motors in order to obtain arbitrary poses of the end-effector platform with respect to the fixed platform.

Let XYZ be a reference frame attached to the center of the fixed platform. The motors M_4, M_5 and M_6 have the function to control the position of the center of the spherical joint connecting the middle platform to the end-effector platform, the point $P=(p_X, p_Y, p_Z)$ located by vector $\boldsymbol{P}$. To this end, the generalized coordinates q_4, q_5 and q_6 are related, respectively, with the *X, Y* and *Z* axes, i.e., $P=(q_4, q_5, q_6)$. On the other hand, the motors M_4, M_5 and M_6 are devoted to control the orientation of the end-effector platform; the associated generalized coordinates are noted as q_1, q_2 and q_3.

In what follows the kinematic analyses of the translational and spherical parallel manipulators are provided.

3. The translational manipulator

As shown in Fig. 1 the chosen translational manipulator, a variant of the mechanism introduced by Gallardo-Alvarado *et al.* (2012), for this application consists mainly of a fixed platform, a cross

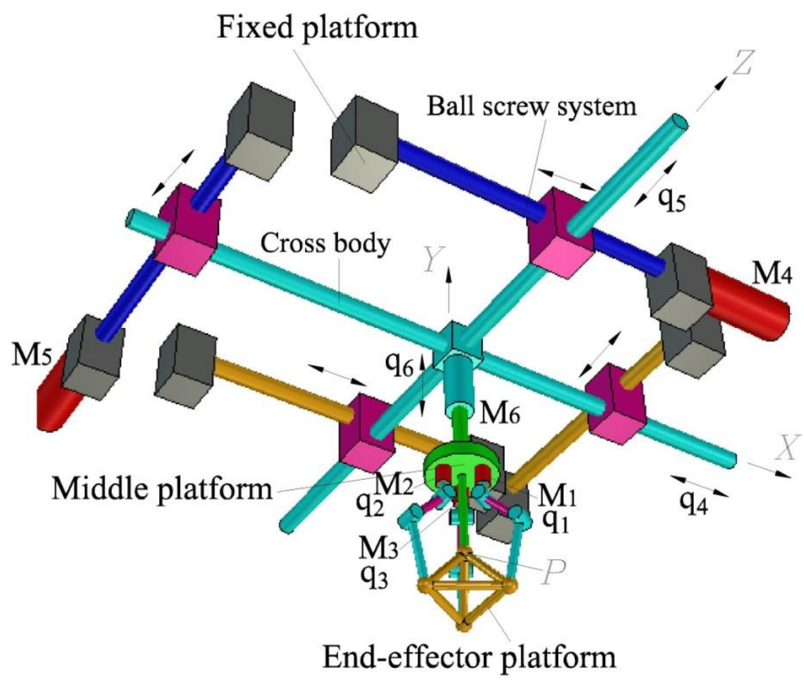

Fig. 1 The proposed series-parallel manipulator

body and a middle platform. The cross body is connected to the fixed platform through two ball screw systems and two double prismatic joints. A double prismatic joint is a passive element where the directions of the two prismatic joints are mutually orthogonal. Furthermore, the cross body is connected to the end-effector platform by means of an active prismatic joint. It is straightforward to demonstrate that owing the decoupled orthogonal arrangement of the generalized coordinates q_4, q_5 and q_6, it is possible to easily establish the following Input/Output displacement relationship

$$\mathbf{A}_t P = \mathbf{B}_t \mathbf{q}_t \tag{1}$$

where $\mathbf{A}_t$=$\mathbf{B}_\mathrm{t}$ is the identity matrix of order 3 and $\mathbf{q}_\mathrm{t}=[q_4\ q_5\ q_6]^T$ is the *translational displacement first-order driver matrix*. This follows that the velocity and acceleration analyses are straightforward. Furthermore, taking into account that each related Jacobian given in expression (1) is precisely the identity matrix, then the translational manipulator hand is a fully-isotropic parallel manipulator, i.e., the condition number equals 1 at any posture of the mechanism. On the other hand, due to the simplicity of the Input/Output relationships, see Eq. (1), the theoretical workspace, i.e., the locus of the points that the reference point P on the end-effector platform can reach, is directly found to be a parallelepiped of edges $q_i(i=4, 5, 6)$.

Concluding this brief section, it is demonstrated that the translational manipulator is fully-decoupled, fully-isotropic, singularity-free in the entire workspace and possesses the maximum available workspace. Even though that the translational manipulator considered in this section is an asymmetrical manipulator, it can be considered as a subclass of TPMs with linear Input/Output equations (Kong and Gosselin 2002, Kong and Gosselin 2004, Gosselin *et al*. 2004).

4. The parallel wrist

A typical spherical parallel manipulator, also known as a parallel wrist, consists of a moving platform and a fixed platform connected each other by means of three limbs provided with multiple revolute joints whose axes intersect at a common point, see for instance Cox 1981, Di Gregorio (2001, 2004). The complexity of such mechanical arrangement is evident. In fact, in order to achieve successfully the intersection of such axes, higher manufacturing conditions must be satisfied otherwise jamming or higher internal loads emerge that can damage the robot. In that concern it is worth to note that the intersection of revolute axes is a mandatory condition not only for most parallel wrists but also for four-five degrees-of-freedom parallel manipulators, see for instance Zlatanov and Gosselin (2001), Li *et al*. (2004), Zhu *et al*. (2009). This overconstrained condition may be eliminated by connecting the moving and fixed platforms through a fixed spherical joint which plays the role of the fixed rotation center Innocenti and Parenti-Castelli (1993), Wohlhart (1994), Alici and Shirinzadeh (2004). After, the limbs may be designed freely.

4.1 Description of the parallel wrist

With the purpose to enhance the advantages of the selected parallel wrist, a mechanism introduced by Gallardo-Alvarado *et al.* (2013), for the series-parallel manipulator it is advisable firstly to recall some typical spherical parallel manipulators reported in the literature. The 3-RRR parallel wrist is a symmetric manipulator with revolute joints whose axes intersect at a common point, namely the rotation center of the manipulator, in order to achieve spherical motions. This parallel manipulator has been extensively studied. Certainly, some relevant kinematic properties of the 3RRR parallel wrist had been deeply investigated, and not *diffusely* as it is pointed in Di Gregorio (2007), such as displacement analysis, optimum kinematic design, singularity analysis, control and so forth, for detailed information the reader is referred to Gosselin and Angeles (1988), Gosselin *et al*. (1996), Liu *et al*. (2000), Gosselin and Wang (2002), Bonev and Gosselin (2006), Bonev *et al*. (2006), Bai *et al*. (2009). It is worth to mention that the most celebrated spherical parallel manipulator based on a 3RRR topology is the famous robot called Agile Eye developed at Laval University. On the other hand, the 3S{P,R}S+S parallel wrist is another option concerning with the spherical motion of a rigid body. This robot consists of a moving platform and a fixed platform connected each other through a passive constrained spherical joint, locating the rotation center of the manipulator, and three S{P,R}S-type limbs, its forward displacement analysis was addressed successfully by Innocenti and Parenti-Castelli (1993), who conclude that there are at most eight distinct orientations that the moving platform can reach with respect to the fixed platform given the limb lengths of the manipulator. This compact in-parallel manipulator brings interesting characteristics such as the possibility to use linear actuators when the version 3SPS+S is chosen; moreover, clearly a non-overconstrained parallel manipulator, like this, support manufacturing errors due to the use of spherical joints instead of revolute joints. Certainly, in a glance it seems that a 3S{P,R}S+S parallel manipulator has poor manipulability and reduced workspace when it is compared with the 3RRR parallel wrist, however it is worth to mention that Alici and Shirizandeh (2004) proved that this parallel wrist possesses, in reality, a considerable workspace with the possibility to alleviate singularities when a topology optimization method is employed. With these considerations in mind, the spherical parallel manipulator considered in this work is a variant of the 3S{P,R}S+S parallel wrist.

The parallel wrist of the contribution, see Fig. 2, is a 3$\underline{\text{R}}$RRS+S spherical parallel manipulator.

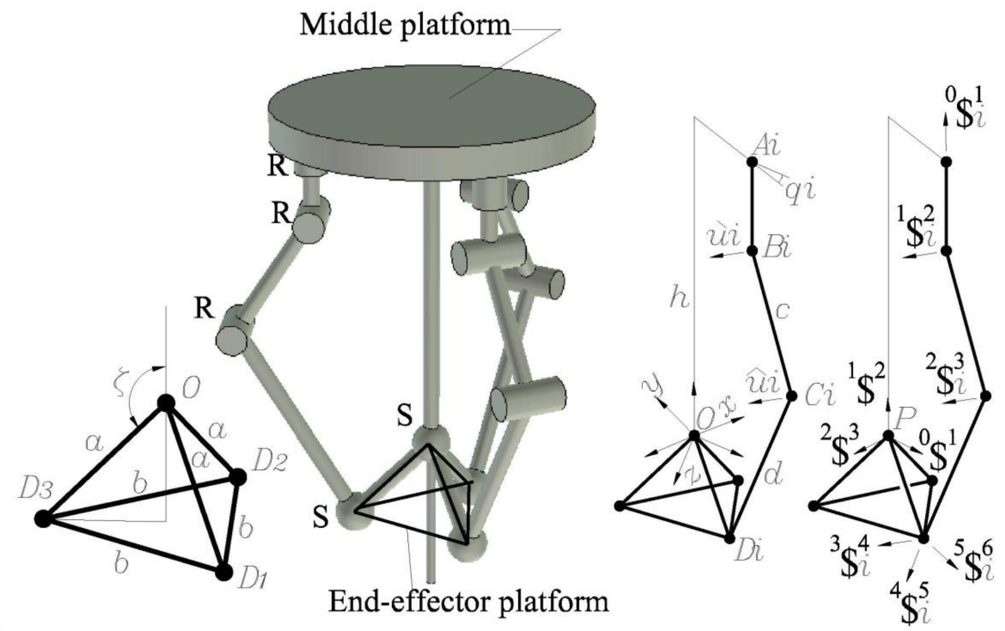

Fig. 2 The parallel wrist

A triangular fixed platform ($A_1 A_2 A_3$), labeled 0, is connected to an orthogonal moving platform ($OD_1 D_2 D_3$), labeled *m* also known as the end-effector platform, through a passive constrained spherical joint, located at origin O in this case $O = P$ of the fixed reference frame, and three identical limbs. The orientation of the moving platform is controlled by means of three actuators R mounted on the middle platform. The axes of the actuated revolute joints are parallel to each other and perpendicular to the axes of the passive revolute joints. Furthermore, the 3RRRS+S parallel wrist is free of revolute joints with axes intersecting at a common point, the main advantage of this robot. Owing symmetry of the moving platform, its link parameters are $\|OD_i\|=a$ and $\|D_iD_j\|=b$ (i, j=1, 2, 3 mod(3)). Finally, in order to simplify the analysis, the reference frame XYZ and the moving reference frame xyz, attached respectively to platforms 0 and *m*, share a common origin O=P.

4.2 Displacement analysis of the parallel wrist

In this section the displacement analysis of the spherical parallel manipulator is presented. For simplicity and without loss of generality consider that the reference frame XYZ is located at the center of the spherical joint connecting the end-effector platform to the middle platform. The forward position analysis consists of finding the orientation of the moving platform with respect to the fixed platform given the input joint angles q_i (unless otherwise, in the remainder of the contribution i=1, 2, 3). To this end, consider that passive revolute joints affect the displacement of the center of the spherical joint in the same limb in such a way that

$$(\boldsymbol{B}_i - \boldsymbol{D}_i) \bullet \hat{\boldsymbol{u}}_i = 0 \tag{2}$$

where $\hat{\boldsymbol{u}}_i = \sin(q_i)\,\hat{i} + 0\,\hat{j} + \cos(q_i)\,\hat{k} = ux\hat{i} + uz\hat{k}$ is the unit vector along the axes of the passive revolute joints in the same limb in accordance with the $\hat{i}, \hat{j}, \hat{k}$ unit vectors of the axes X, Y, Z of the fixed reference frame, and the bullet ($\bullet$) denotes the inner product of the usual three-dimensional vector algebra. Hence, considering that $D_i=(X_i, Y_i, Z_i)$ are the unknown coordinates associated to vector $\boldsymbol{D}_i$ whereas $B_i=(Bx_i, By_i, Bz_i)$are the coordinates associated to vector $\boldsymbol{B}_i$ then the unknowns X_i are obtained based on Eq. (2) as follows

$$X_i = \tilde{A}_i Z_i + \tilde{B}_i \tag{3}$$

where $A_i = -uz/ux_i$ and $\tilde{B}_i = (Bx_i ux_i + uz_i Bz_i)/ux_i$. Furthermore, it is evident that closure equations for parameter a may be expressed as

$$\boldsymbol{D}_i \bullet \boldsymbol{D}_i = a^2 \tag{4}$$

Thus, the substitution of Eq. (3) into Eq. (4) allows to express the unknowns Y_i as

$$Y_i = \sqrt{-(1+\tilde{A}_i^2)Z_i^2 - 2\tilde{A}_i\tilde{B}_i Z_i + a^2 - \tilde{B}_i^2} \tag{5}$$

Finally, consider the following closure equations associated to parameter b

$$(\boldsymbol{D}_i - \boldsymbol{D}_j)\bullet(\boldsymbol{D}_i - \boldsymbol{D}_j) = b^2 \quad i, j = 1, 2, 3 \mod(3) \tag{6}$$

Later on, the substitution of Eqs. (3) and (5) into Eq. (6) yields a nonlinear system of three equations in the unknowns Z_1, Z_2, and Z_3 as

$$\begin{gathered}\tilde{C}_{ij} Z_i^2 Z_j^2 + \tilde{D}_{ij} Z_i^2 Z_j + \tilde{E}_{ij} Z_i Z_j^2 + \tilde{F}_{ij} Z_i^2 + \tilde{G}_{ij} Z_j^2 + \tilde{H}_{ij} Z_i Z_j + \\ \tilde{I}_{ij} Z_i + \tilde{J}_{ij} Z_j + \tilde{K}_{ij} = 0 \quad i, j = 1, 2, 3 \mod(3)\end{gathered} \tag{7}$$

where the coefficients of Eq. (7) are all functions of the generalized coordinates and parameters of the parallel wrist. In fact

$$\tilde{C}_{ij} = -4(\tilde{A}_i - \tilde{A}_j)^2$$
$$\tilde{D}_{ij} = 8\tilde{B}_j(\tilde{A}_i - \tilde{A}_j)$$
$$\tilde{E}_{ij} = -8\tilde{B}_i(\tilde{A}_i - \tilde{A}_j)$$
$$\tilde{F}_{ij} = 4a^2(1+\tilde{A}_i^2) - 4\tilde{B}_j^2$$
$$\tilde{G}_{ij} = 4a^2(1+\tilde{A}_j^2) - 4\tilde{B}_i^2$$
$$\tilde{H}_{ij} = 4b^2 - 8a^2 + 8\tilde{B}_i\tilde{B}_j + 4\tilde{A}_i\tilde{A}_j(b^2 - 2a^2)$$
$$\tilde{I}_{ij} = 8\tilde{A}_i\tilde{B}_i a^2 + 4\tilde{A}_i\tilde{B}_j(b^2 - 2a^2)$$
$$\tilde{J}_{ij} = 8\tilde{A}_j\tilde{B}_j a^2 + 4\tilde{A}_j\tilde{B}_i(b^2 - 2a^2)$$
$$\tilde{K}_{ij} = b^4 + 4a^2(\tilde{B}_i^2 + \tilde{B}_j^2 - b^2) - 4\tilde{B}_i\tilde{B}_j(2a^2 - b^2)$$

Expressions (7) are reduced systematically into an 8-*th* order univariate polynomial equation by resorting to the Sylvester's dialytic method of elimination, for details the reader is referred to Tsai (1999), Merlet (1989), Gallardo-Alvarado *et al*. (2008). Once the coordinates of points D_i are calculated, the orientation of the moving platform is determined from the computation of the rotation matrix ${}^{0}\mathbf{R}^{m}$. In fact, it is possible to form a linear system of nine equations in the nine unknown elements of the rotation matrix taking into proper account that

$$D_i = {}^{0}\mathbf{R}^{m} D_i' \tag{8}$$

where

$${}^{0}\mathbf{R}^{m} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{9}$$

so that D_i' are the coordinates of the center of the *i-th* spherical joint, as expressed in the reference frame xyz. Furthermore, see Craig (1955), the roll, pitch and yaw angles are computed, respectively, as

$$\left.\begin{aligned} \gamma &= \arctan\left(r_{32}/r_{33}\right) \\ \beta &= \arctan\left(-r_{31}/\sqrt{r_{11}^2 + r_{21}^2}\right) \\ \alpha &= \arctan\left(r_{21}/r_{11}\right) \end{aligned}\right\} \tag{10}$$

Finally, the points C_i, located by vectors $\boldsymbol{C}_i$, are obtained taking into proper account the closure equations

$$\left.\begin{aligned} (\boldsymbol{C}_i - \boldsymbol{B}_i) \bullet (\boldsymbol{C}_i - \boldsymbol{B}_i) &= c^2 \\ (\boldsymbol{C}_i - \boldsymbol{D}_i) \bullet (\boldsymbol{C}_i - \boldsymbol{D}_i) &= d^2 \\ (\boldsymbol{C}_i - \boldsymbol{B}_i) \bullet \hat{\boldsymbol{u}}_i &= 0 \end{aligned}\right\} \tag{11}$$

On the other hand, the inverse position analysis is concerned with determining the actuated-joint angles q_i given the orientation of the moving platform with respect to the fixed platform. This analysis is somewhat trivial from a mathematical point of view; in fact, consider as an intermediate step that the coordinates of points D_i can be obtained easily, upon the known rotation matrix ${}^{0}\mathbf{R}^{m}$, as

$$D_i = {}^{0}\mathbf{R}^{m} D_i' \tag{12}$$

where D_i are the coordinates of the center of the *i-th* spherical joint as expressed in the moving reference frame xyz. After that, the angles q_i are obtained by resorting to Eq. (2).

5. Infinitesimal kinematics of the parallel wrist

In this section the velocity and acceleration analyses of the spherical parallel manipulator are addressed by means of the screw theory.

5.1 Input/Output equation of velocity

Let ${}^{0}\boldsymbol{\omega}^{m}$ and ${}^{0}\boldsymbol{v}_O^{m} = \mathbf{0}$ be the angular and linear velocity vectors of body m as measured from body 0 taking $O=P$ as the reference pole. The velocity state between bodies m and 0, the six-dimensional vector ${}^0\boldsymbol{V}_O^m = \begin{bmatrix} {}^0\boldsymbol{\omega}^m \\ {}^0\boldsymbol{v}_O^m \end{bmatrix} = \begin{bmatrix} {}^0\boldsymbol{\omega}^m \\ \mathbf{0} \end{bmatrix}$ may be written through the RRRS-type limbs of the parallel wrist as a linear combination of the joint-rate velocities in the same leg as follows

$$\dot{q}_i \, {}^0\$_i^1 + {}_1\omega_2^i \, {}^1\$_i^2 + \ldots + {}_4\omega_5^i \, {}^4\$_i^5 + {}_5\omega_6^i \, {}^5\$_i^6 = {}^0\boldsymbol{V}_O^m \tag{13}$$

or

$$\mathbf{J}_i \boldsymbol{\Omega}_i = {}^0\boldsymbol{V}_O^m \tag{14}$$

where

$\mathbf{J}_i = \begin{bmatrix} {}^0\$_i^1 & {}^1\$_i^2 & {}^2\$_i^3 & {}^3\$_i^4 & {}^4\$_i^5 & {}^5\$_i^6 \end{bmatrix}$ is the screw-coordinate Jacobian matrix of the *i-th* limb

$\boldsymbol{\Omega}_i = \begin{bmatrix} \dot{q}_i & {}_1\omega_2^i & {}_2\omega_3^i & {}_3\omega_4^i & {}_4\omega_5^i & {}_5\omega_6^i \end{bmatrix}^T$ is the matrix containing the joint-rate velocities in the *i-th* limb

Please note that the screw ${}^3\$_i^4$ is reciprocal to all the screws, excepting ${}^0\$_i^1$, in the same limb. Thus, the application of the Klein form $\{*;*\}$[1] of ${}^3\$_i^4$ to both sides of Eq. (13), with the reduction of terms, leads to

$$\left\{ {}^3\$_i^4 ; \dot{q}_i \, {}^0\$_i^1 \right\} = \left\{ {}^3\$_i^4 ; {}^0\boldsymbol{V}_O^m \right\} \tag{15}$$

After a few computations, it follows that the I/O velocity equation of the parallel wrist is given by

$$\mathbf{A} \, {}^0\boldsymbol{\omega}^m = \mathbf{B}\,\mathbf{v} \tag{16}$$

where

$\mathbf{A} = \begin{bmatrix} D({}^3\$_1^4) & D({}^3\$_2^4) & D({}^3\$_3^4) \end{bmatrix}$

$\mathbf{B} = \begin{bmatrix} \{{}^3\$_1^4 ; {}^0\$_1^1\} & 0 & 0 \\ 0 & \{{}^3\$_2^4 ; {}^0\$_2^1\} & 0 \\ 0 & 0 & \{{}^3\$_3^4 ; {}^0\$_3^1\} \end{bmatrix}$

$\mathbf{v} = \begin{bmatrix} \dot{q}_1 & \dot{q}_2 & \dot{q}_3 \end{bmatrix}^T$

Likewise, the vector ${}^0\boldsymbol{\omega}^m$ can be obtained considering that resorting to the central spherical joint it is possible to write an auxiliary vector equation as

$${}_0\omega_1 \, {}^0\$^1 + {}_1\omega_2 \, {}^1\$^2 + {}_2\omega_3 \, {}^2\$^3 = {}^0\boldsymbol{V}_O^m \tag{17}$$

After, since the screws ${}^0\$^1$, ${}^1\$^2$ and ${}^2\$^3$ are reciprocal to each other, then it is evident that

$$\left\{ {}^0\$^1 ; {}^0\boldsymbol{V}_O^m \right\} = \left\{ {}^1\$^2 ; {}^0\boldsymbol{V}_O^m \right\} = \left\{ {}^2\$^3 ; {}^0\boldsymbol{V}_O^m \right\} = 0 \tag{18}$$

Casting into a matrix-vector form expressions (15) and (18) it follows that the Input/Output

---

[1]Let $\$_1 = (\boldsymbol{s}_1, \boldsymbol{s}_{O1})$ and $\$_2 = (\boldsymbol{s}_2, \boldsymbol{s}_{O2})$ be two elements of the Lie algebra *se*(3), which is isomorphic to the screw theory, of the Euclidian group *SE*(3). Then, the Klein form, denoted as $\{* ; *\}$, is defined as $\{\$_1 ; \$_2\} = \boldsymbol{s}_1 . \boldsymbol{s}_{O1} + \boldsymbol{s}_2 . \boldsymbol{s}_{O2}$

equation of velocity is also given by

$$\mathbf{J}^T \boldsymbol{\Delta}\ {}^0\mathbf{V}_O^m = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix} \tag{19}$$

where

$\mathbf{J} = \left[{}^3\$_1^4 \ \ {}^3\$_2^4 \ \ {}^3\$_3^4 \ \ {}^0\$^1 \ \ {}^1\$^2 \ \ {}^2\$^3 \right]$ is the active screw-coordinate Jacobian matrix of the manipulator

$\boldsymbol{\Delta} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}$ is a partitioned matrix defined by the zero matrix **0** and the identity matrix **I** that plays the role of an operator of polarity, Lipkin and Duffy 1985.

5.2 Input/Output equation of acceleration

Let ${}^0\dot{\boldsymbol{\omega}}^m$ and ${}^0\boldsymbol{a}_O^m = \mathbf{0}$ be the angular and linear acceleration vectors of body *m* with respect to body 0, where the origin *O*=*P* is the reference pole. The reduced acceleration state between bodies *m* and 0, the six-dimensional vector ${}^0\boldsymbol{A}_O^m = \begin{bmatrix} {}^0\dot{\boldsymbol{\omega}}^m \\ {}^0\boldsymbol{a}_O^m - {}^0\boldsymbol{\omega}^m \times {}^0\boldsymbol{v}_O^m \end{bmatrix} = \begin{bmatrix} {}^0\dot{\boldsymbol{\omega}}^m \\ \mathbf{0} \end{bmatrix}$, may be written through the RRRS-type limbs as follows

$$\ddot{q}_i\ {}^0\$_i^1 + {}_1\dot{\omega}_2^i\ {}^1\$_i^2 + \ldots + {}_4\dot{\omega}_5^i\ {}^4\$_i^5 + {}_5\dot{\omega}_6^i\ {}^5\$_i^6 + \mathcal{L}_i = {}^0\boldsymbol{A}_O^m \tag{20}$$

or

$$\mathbf{J}_i \dot{\boldsymbol{\Omega}}_i + \mathcal{L}_i = {}^0\boldsymbol{A}_O^m \tag{21}$$

where $\dot{\boldsymbol{\Omega}}_i = \left[\ddot{q}_i \ \ {}_1\dot{\omega}_2^i \ \ {}_2\dot{\omega}_3^i \ \ {}_3\dot{\omega}_4^i \ \ {}_4\dot{\omega}_5^i \ \ {}_5\dot{\omega}_6^i \right]^T$ is a matrix containing the joint-rate accelerations of the *i-th* limb and $\mathcal{L}_i$ is the *i-th* Lie screw of acceleration which are computed as

$$\begin{aligned} \mathcal{L}_i = &\left[\dot{q}_i\ {}^0\$_i^1 \quad {}_1\omega_2^i\ {}^1\$_i^2 + \ldots + {}_4\omega_5^i\ {}^4\$_i^5 + {}_5\omega_6^i\ {}^5\$_i^6 \right] \\ &+ \left[{}_1\omega_2^i\ {}^1\$_i^2 \quad {}_2\omega_3^i\ {}^2\$_i^3 + {}_3\omega_4^i\ {}^3\$_i^4 + {}_4\omega_5^i\ {}^4\$_i^5 + {}_5\omega_6^i\ {}^5\$_i^6 \right] \\ &+ \left[{}_2\omega_3^i\ {}^2\$_i^3 \quad {}_3\omega_4^i\ {}^3\$_i^4 + {}_4\omega_5^i\ {}^4\$_i^5 + {}_5\omega_6^i\ {}^5\$_i^6 \right] \\ &+ \left[{}_3\omega_4^i\ {}^3\$_i^4 \quad {}_4\omega_5^i\ {}^4\$_i^5 + {}_5\omega_6^i\ {}^5\$_i^6 \right] + \left[{}_4\omega_5^i\ {}^4\$_i^5 \quad {}_5\omega_6^i\ {}^5\$_i^6 \right] \end{aligned} \tag{22}$$

where the brackets $[* \ *]$ denote the lie product of the Lie algebra *se*(3) of the Euclidean group *SE*(3). Furthermore, through the central spherical joint, the accelerator ${}^0\boldsymbol{A}_O^m$ may be obtained as

$${}^0\boldsymbol{A}_O^m = {}_0\dot{\omega}_1\ {}^0\$^1 + {}_1\dot{\omega}_2\ {}^1\$^2 + {}_2\dot{\omega}_3\ {}^2\$^3 \tag{23}$$

Following the trend of the velocity analysis, the Input/Output equation of acceleration results in

$$\mathbf{A}\,{}^0\dot{\boldsymbol{\omega}}^m = \mathbf{B}\mathbf{a} + \mathbf{C} \tag{24}$$

where

$\mathbf{a} = \left[\ddot{q}_1 \ \ \ddot{q}_2 \ \ \ddot{q}_3 \right]^T$

$\mathbf{C} = \left[\left\{ {}^3\$_1^4; \mathcal{L}_1 \right\} \ \ \left\{ {}^3\$_2^4; \mathcal{L}_2 \right\} \ \ \left\{ {}^3\$_3^4; \mathcal{L}_3 \right\} \right]^T$ is the complementary matrix of acceleration.

Whereas according to the central spherical joint, the Input/Output equation of acceleration of the parallel wrist may be obtained as

$$\mathbf{J}^T \boldsymbol{\Delta}\ {}^0\!\boldsymbol{A}_O^m = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{C} \\ \mathbf{0} \end{bmatrix} \tag{25}$$

6. Case study

In this section a numerical example concerned with the forward kinematics of the robot is provided. Conveniently, the numerical example is subdivided into two parts: i) kinematics of the translational manipulator, ii) kinematics of the parallel wrist. Furthermore, SI units are used thorough the case study.

6.1 Forward kinematics of the translational manipulator

In the home position of the series-parallel manipulator the center of the movable cross body is located at the origin of the XYZ reference frame. Furthermore, consider that the corresponding generalized coordinates are commanded to follow periodical functions given by q_4=0.5sin(t)cos(t), q_5=-0.75-0.5sin(t)cos(t) and q_6=0.75sin(t)cos(t) where the interval for time t is given by $0 \le t \le 2\pi$. Hence the position of point P with respect to the origin of the fixed reference frame XYZ results in P=(0.5sin(t)cos(t), -0.75-0.5sin(t)cos(t), 0.75sin(t)cos(t)). After, the computation of the velocity and acceleration of point P is straightforward. The benefits of a fully-decoupled mechanism are evident.

6.2 Forward kinematics of the spherical parallel manipulator

The values for the basic parameters of the parallel wrist are chosen as a=0.1414, b=0.1732, c=0.1803, d=0.2915, e=0.3354, A_1=(0.15, 0.4, 0), A_2=(-0.075, 0.4, -0.1299), A_3=(-0.075, 0.4, 0.1299), B_1=(0.15, 0.3, 0), B_2=(-0.075, 0.3, -0.1299), and B_3=(-0.075, 0.3, 0.1299). Furthermore, at the time t=0 the unit vectors along the axes of the passive revolute joints are selected as: $\hat{\boldsymbol{u}}_1 = \hat{k}$, $\hat{\boldsymbol{u}}_2 = 0.866\hat{i} - 0.5\hat{k}$, $\hat{\boldsymbol{u}}_3 = 0.866\hat{i} + 0.5\hat{k}$. With these data, there are three resulting available real solutions for the forward displacement analysis. These are listed in Table 1.

The next part of the exercise requires the application of the formulae developed to solve the forward infinitesimal kinematics of the parallel wrist. Let solution 1 of Table 1 be the home

Table 1 Real solutions for the forward displacement analysis of the parallel wrist

Sol.	Coordinates
1	D_1=(0.099, -0.1, 0) D_2=(-0.05, -0.099, -0.086) D_3=(-0.05, -0.098, 0.086)
2	D_1=(-0.1, -0.099, 0) D_2=(-0.069, 0.022, -0.121) D_3=(0.05, -0.098, -0.086)
3	D_1=(0.099, -0.1, 0)

D_2=(-0.05, -0.099, -0.086) D_3=(0.069, -0.063, -0.121)

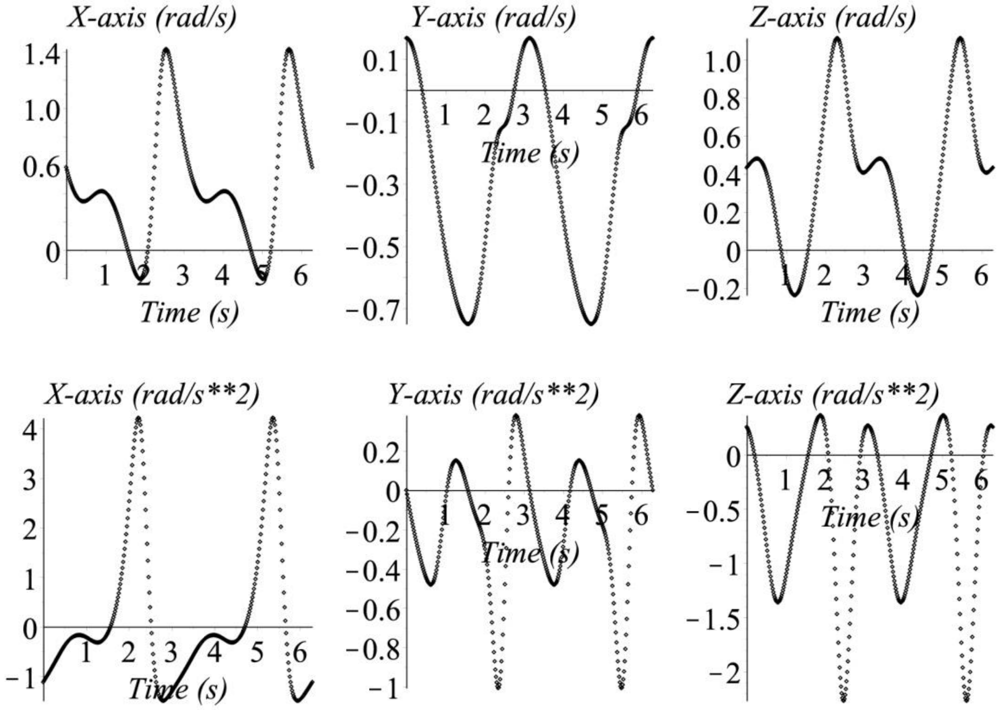

Fig. 3 Time history of the components of the angular velocity and acceleration of the end-effector platform as measured from the fixed platform

position of the robot, or in other words the configuration of the spherical parallel manipulator at the time t=0. Moreover, consider that the input joint angles are governed by periodical functions given by q_1=-1.5sin(t)cos(t), q_2=2.0944+sin(t)cos(t), and q_3=1.0472-0.5sin(t)cos(t) in the interval $0 \leq t \leq 2\pi$. After, the corresponding resulting temporal behavior of the angular velocity and acceleration of the end-effector platform is provided in Fig. 3.

7. Conclusions

It is well known that the forward displacement analysis of the general Gough platform, a non-redundant six-legged parallel manipulator introduced almost six decades ago, is one of the most challenging and complicated problems of modern kinematics. The development of series-parallel manipulator is a viable option to ameliorate such problem preserving some benefits of the original hexapod.

In this work a series-parallel manipulator built with two different robot manipulators is introduced. The first mechanism is a fully-decoupled, fully-isotropic and singularity-free translational manipulator while the second manipulator is a parallel wrist with partially decoupled kinematics equipped with rotational generalized coordinates with the advantage that the parallel wrist is free of revolute joints with axes intersecting at a common point. Hence, taking into account that a decoupled manipulator is a robot where the orientation of the end-effector platform may be computed disregarding the corresponding position, the proposed robot is decoupled series-parallel manipulator. Simple kinematics and control are the main benefits of the proposed series-parallel manipulator. Finally, in order to exemplify the method of kinematic analysis, a numerical example is included.

Acknowledgments

This work has been supported by DGEST and Conacyt of México.

References

Alici, G. and Shirinzadeh, B. (2004), "Topology optimisation and singularity analysis of a 3-SPS parallel manipulator with a passive constraining spherical joint", *Mech. Mach. Theory*, **39**, 215-235.

Altuzarra, O., Loizaga, M., Pinto, C. and Petuya, V. (2010), "Synthesis of partially decoupled multi-level manipulators with lower mobility", *Mech. Mach. Theory*, **45**(1), 106-118.

Bai, S., Hansen, M.R. and Angeles, J. (2009), "A robust forward-displacement analysis of spherical parallel robots", *Mech. Mach. Theory*, **44**(12), 2204-2216.

Bonev, I.A. and Gosselin, C.M. (2006), "Analytical determination of the workspace of symmetrical spherical parallel mechanisms", *IEEE T. Robot.*, **22**(5), 1011-1017.

Bonev, I.A., Chablat, C. and Wenger, P. (2006), "Working and assembly modes of the Agile Eye", *Proceedings of the 2006 IEEE International Conference on Robotics and Automation*, Orlando, Florida, USA.

Cox, D. (1981), "The dynamic modeling and command signal formulation for parallel multi-parameter robotic devices", Ph.D. Dissertation, University of Florida.

Craig, J.J. (1955), *Introduction to Robotics: Mechanics & Control*, Addison-Wesley Publishing Company.

Di Gregorio, R. (2001), "A new parallel wrist employing just revolute pairs: the 3-RUU wrist", *Robotica*, **19**(3), 305-309.

Di Gregorio, R. (2004), "The 3-RRS wrist: a new, simple and non-overconstrained spherical parallel manipulator", ASME, *J. Mech. Des.*, **126**(5), 850-855.

Di Gregorio, R. (2007), "Parallel wrists: limb architectures and mobility analysis", *2nd EURON Winter School - Parallel Robots: Theory and Applications EURON 2007*, Benidorm, Spain.

Gallardo, J., Orozco, H. and Rico, J.M. (2010), "A novel five-degrees-of-freedom decoupled robot", *Robotica*, **28**(6), 909-917.

Gallardo-Alvarado, J., Aguilar-Nájera, C.R., Casique-Rosas, L., Rico-Martínez, J.M. and Nazrul Islam, Md (2008), "Kinematics and dynamics of 2(3-RPS) manipulators by means of screw theory and the principle of virtual work", *Mech. Mach. Theory*, **43**(10), 1281-1294.

Gallardo-Alvarado, J., García-Murillo, M.A. and Pérez-González, L. (2013), "Kinematics of the 3RRRS+S parallel wrist", *Mech. Base. Des. Struct. Mach.*, **41**, 452-467.

Gallardo-Alvarado, J., Orozco-Mendoza, H., Sánchez-Rodríguez, A. and Alici, G. (2012), "Kinematic analyses of novel translational parallel manipulators", *Proceeding of the IMechE Part C: J. Mechanical Eng. Sci.*, **228**(2), 330-341.

Gallardo-Alvarado, J., Rodríguez-Castro, R. and Islam, Md, N. (2008), "Analytical solution of the forward position analysis of parallel manipulators that generate 3-RS structures", *Adv. Robot.*, **22**(2-3), 215-234.

Gan, D., Liao, Q., Dai, J.S., Wei, S. and Seneviratne, L.D. (2009), "Forward displacement analysis of the general 6-6 Stewart mechanism using Gröbner bases", *Mech. Mach. Theory*, **44**(9), 1640-1647.

Gosselin, C.M., Kong, X., Foucault, S. and Bonev, I.A. (2004), "A fully decoupled 3-DOF translational parallel mechanism", *Proceedings 4th Chemnitz Parallel Kinematics Seminar, Parallel Kinematic Machines International Conference*, Chemnitz, Germany.

Gosselin, C. and Angeles, J. (1988), "The optimum kinematic design of a planar three-degree-of-freedom parallel manipulator", ASME, *J. Mech. Transm. Autom. Des.*, **110**(1), 35-41.

Gosselin, C., St-Pierre, E. and Gagné, M. (1996), "On the development of the Agile Eye: mechanical design, control issues and experimentation", *IEEE Robot. Automat. Mag.*, **3**(4), 29-37.

Gosselin, C.M. and Wang, J. (2002), "Singularity loci of a special class of spherical three-degree-of-freedom parallel mechanisms with revolute actuators", *Int. J. Robot. Res.,* **21**(7), 649-659.

Gough, V.E. (1957), "Contribution to discussion of papers on research in automobile stability", *Proceedings of the automobile division of the institution of mechanical engineers*, New York.

Gough, V.E. and Whitehall, S.G. (1962), "Universal tire testing machine", *Proceedings of the 9th international automobile technical congress, Federation Internationale des Societes d'Ingenieurs des Techniques de l'Automobile (FISITA), IMechE 1,* London, UK.

Husty, M.L. (1996), "An algorithm for solving the direct kinematics of general Stewart-Gough platforms", *Mech. Mach. Theory*, **31**(4), 365-379.

Innocenti, C. (2001), "Forward kinematics in polynomial form of the general Stewart platform", ASME, *J. Mech. Des.*, **123**(2), 254-260.

Innocenti, C. and Parenti-Castelli, V. (1993), "Echelon form solution of direct kinematics for the general fully-parallel spherical wrist", *Mech. Mach. Theory*, **28**(4), 553-561.

Kong, X. and Gosselin, C.M. (2002), "Kinematics and singularity analysis of 3-CRR 3-DOF translational parallel manipulators", *Int. J. Robot. Res.*, **21**(9), 791-798.

Kong, X. and Gosselin, C.M. (2004), "Type synthesis of 3-DOF translational parallel manipulators based on screw theory", ASME, *J. Mech. Des.*, **126**(1), 83-92.

Legnani, G., Fassi, I., Giberti, H., Cinquemani, S. and Tosi, D. (2012), "A new isotropic and decoupled 6-DoF parallel manipulator", *Mech. Mach. Theory*, **58**, 64-81.

Li, Q.C., Huang, Z. and Herve, J.M. (2004), "Type synthesis of 3R2T 5-DoF parallel manipulators using the Lie group of displacements", *IEEE T. Robotic. Automat.*, **20**(2), 173-180.

Liu, X.J., Jin, Z.L. and Gao, F. (2000), "Optimum design of 3-DOF spherical parallel manipulators with respect to the conditioning and stiffness indices", *Mech. Mach. Theory*, **35**(9), 1257-1267.

Lipkin, H. and Duffy, J. (1985), "The elliptic polarity of screws", ASME, *J. Mech. Transm. Autom. Des.*, **107**, 377-387.

Merlet, J.P. (1989), Manipulateurs paralleles, 4eme partie: mode d'assemblage et cinematique directe sous forme polynomial, INRIA Research Report no. 1135, Centre de Sophia Antipolis, Valbonne.

Raghavan, M. (1993), "The Stewart platform of general geometry has 40 configurations", ASME, *J. Mech. Des.*, **115**, 277-282.

Rolland, L. (2005), "Certified solving of the forward kinematics problem with an exact algebraic method for the general parallel manipulator", *Adv. Robotics*, **19**, 995-1025.

Romdhane, L. (1999), "Design and analysis of a hybrid serialparallel manipulator", *Mech. Mach. Theory*, **34**(7), 1037-1055.

Tanev, T.K. (2000), "Kinematics of a hybrid (parallel-serial) robot manipulator", *Mech. Mach. Theory*, **35**, 1183-1196.

Tsai, L.W. (1999), *Robot Analysis*, John Wiley & Sons, New York.

Wampler, C.W. (1996), "Forward displacement analysis of general six-in-parallel SPS (Stewart) platform manipulators using Soma coordinates", *Mech. Mach. Theory*, **31**(3), 331-337.

Wohlhart, K. (1994), "Displacement analysis of the general spherical Stewart platform", *Mech. Mach. Theory*, **29**(4), 581-589.

Zheng, X.Z., Bin, H.Z. and Luo, Y.G. (2004), "Kinematic analysis of a hybrid serial-parallel manipulator", *Int. J. Adv. Manufact. Tech.*, **23**(11-12), 925-930.

Zhu, S.J., Huang, Z. and Zhao, M.Y. (2009), "Singularity analysis for six practicable 5-DoF fully-symmetrical parallel manipulators", *Mech. Mach. Theory*, **44**(4), 710-725.

Zlatanov, D. and Gosselin, C.M. (2001), “A new parallel architecture with four degrees of freedom”, *Proceedings 2nd Workshop on Computational Kinematics*, Seoul, Korea.
Zoppi, M., Zlatanov, D. and Molfino, R. (2006), “On the velocity analysis of interconnected chains mechanisms”, *Mech. Mach. Theory*, **41**(11), 1346-1358.

CC