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A novel technique for removing the rigid body motion in interior BVP of plane elasticity

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Abstract. The aim of this paper is to remove the rigid body motion in the interior boundary value problem (BVP) of plane elasticity by solving the interior and exterior BVPs simultaneously. First, we formulate the interior and exterior BVPs simultaneously. The tractions applied on the contour in two problems are the same. After adding and subtracting the two boundary integral equations (BIEs), we will obtain a couple of BIEs. In the coupled BIEs, the properties of relevant integral operators are modified, and those integral operators are generally invertible. Finally, a unique solution for boundary displacement of interior region can be obtained.

Keywords: complex variable boundary integral equation; interior boundary value problem; neumann boundary value problem; removal of rigid body motion

1. Introduction

In the boundary integral equation (BIE) method of plane elasticity, we will meet two kinds of non-unique solution for BIE. If one considers the boundary value problem after discretization, the relevant influence matrix may be singular.

The first kind of non-unique solution for BIE can be found in the Neumann boundary value problem (BVP) for interior region. The second kind of non-unique solution for BIE can be found in the Dirichlet boundary value problem, typically, for the exterior region. Under some particular scale which is generally called the degenerate scale, the traction solution in the domain exists even the assumed displacements take the vanishing value along the boundary.

It was pointed out that in the Neumann boundary value problem for interior region in the boundary integral equation of plane elasticity, the solution for displacement is not unique (Blazquez *et al*. 1996). Several techniques were suggested for removal the rigid body motion in the solution. For the Dirichlet boundary value problem of interior region, the conventional boundary integral equation (BIE) is enriched by adding constants and corresponding constraints (Chen *et al*. 2016). Even the used scale is degenerate one, a unique solution can be obtained from the enriched BIE. Degenerate scales appear in the solution of some boundary integral equations (BIE) (Vodička and Petrik 2015). When one uses the degenerate scale as a real scale in derivation or computation in the Dirichlet BVP, the BIE has either multiple solutions or does not have a

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solution. For a general anisotropic material, the solution for the degenerate scale was provided. Some recent works have shown that usage of conformal mapping can lead to exact values of the degenerate scales in plane elasticity (Corfdir and Bonnet 2015).

Degenerate scale for multiply connected Laplace problems was studied (Chen and Shen 2007). In the paper, authors utilize the null-field integral equation to analytically study the degenerate scale problem. A numerical solution was provided for the degenerate scale in antiplane elasticity using the null field boundary integral equation (BIE) (Chen 2016).

A local radial point interpolation method was presented and applied to solid mechanics (Liu *et al*. 2002). A group of meshfree methods based on boundary integral equation have been developed (Gu and Liu 2004). An exact integration for the hypersingular boundary integral equation of two-dimensional elastostatics was suggested (Zhang and Zhang 2020).

For Laplace's equation in circular domains with circular holes, the null field method was suggested. (Lee *et al*. 2015). The stress concentration factor (SCF) along the boundary of a hole and a rigid inclusion in an infinite isotropic solid under the anti-plane shear was studied by using the degenerate kernels (Chen *et al*. 2021a). The stress intensity solution for crack problem was studied by using J-integral and boundary equation method (Chen *et al*. 2021b). Degenerate scale of line segments was derived in this paper (Chen *et al*. 2020) It was pointed that there were two kinds of rank-deficiency problems in the boundary element method (BEM) (Chen *et al*. 2014).

Static analysis of the free-free trusses by using a self-regularization approach was carried out (Chen *et al*. 2018). A Total BETI (TBETI) based domain decomposition algorithm with the preconditioning by a natural coarse grid of the rigid body motions is adapted for the solut ion of contact problems of linear elastostatics (Bouchala *et al*. 2009)

The aim of this paper is to remove the rigid body motion in the interior BVP of plane elasticity by solving interior and exterior BVPs simultaneously. First, we formulate the interior and exterior BVPs simultaneously. The tractions applied on the contour in two problems are the same. After adding and subtracting the two BIEs obtained, we will obtain a couple of BIEs. In the coupled BIEs, the properties of relevant integral operators are modified, and those integral operators are generally invertible. Therefore, a unique solution for the coupled BIEs will be evaluated. Finally, by using this method for the Neumann BVP of an interior region, a unique solution for the displacement is obtainable. The rigid body motion solution for the displacement is removed in the suggested method. Several numerical examples are provided to prove efficiency of the suggested method.

2. Analysis

Analysis presented below mainly depends on two kinds of integral equation. Therefore, the boundary integral equations for interior region and exterior region are introduced.

2.1 Complex variable boundary integral equations for interior region and exterior region

Recently, the complex variable boundary integral equation (CVBIE) for the interior region was suggested (Chen and Wang 2015) (Fig. 1(a))

$$\frac{U(t_o)}{2} + B_1 i \int_{\Gamma} \left(\frac{\kappa - 1}{t - t_o} U(t)dt - L_1(t, t_o)U(t)dt + L_2(t, t_o)\overline{U(t)}dt \right) \tag{1}$$

$$= B_2 i \int_\Gamma \left(2\kappa\, ln|t - t_o|\, Q(t)dt + \frac{t-t_o}{\bar{t}-\bar{t}_o}\overline{Q(t)}d\bar{t}\right), (t_o \in \Gamma)$$

where Γ denotes the boundary of the interior region. In Eq. (1), *U(t)* and *Q(t)* denote the displacement and traction along the boundary, which are defined by

$$U(t) = u(t) + iv(t),\ Q(t) = \sigma_N(t) + i\sigma_{NT}(t), (t \in \Gamma) \tag{2}$$

The definition for $U(t) = u(t) + iv(t)$, $Q(t) = \sigma_N(t) + i\sigma_{NT}(t)$ $(t \in \Gamma)$ can be found from Fig. 1.

In addition, two elastic constants and two kernels are defined by

$$B_1 = \frac{1}{2\pi(\kappa+1)},\quad B_2 = \frac{1}{4\pi G(\kappa+1)} \tag{3}$$

$$L_1(t,\tau) = -\frac{d}{dt}\left\{ln\frac{t-\tau}{\bar{t}-\bar{\tau}}\right\} = -\frac{1}{t-\tau} + \frac{1}{\bar{t}-\bar{\tau}}\frac{d\bar{t}}{dt},\ L_2(t,\tau) = \frac{d}{dt}\left\{\frac{t-\tau}{\bar{t}-\bar{\tau}}\right\} = \frac{1}{\bar{t}-\bar{\tau}} - \frac{t-\tau}{(\bar{t}-\bar{\tau})^2}\frac{d\bar{t}}{dt} \tag{4}$$

where $\kappa = 3 - 4\nu$ (for plane strain condition), $\kappa = (3-\nu)/(1+\nu)$ (for plane stress condition), G is the shear modulus of elasticity, and $\nu$is the Poisson's ratio. In this paper, the plane strain condition and $\nu = 0.3$ are assumed. In Eq. (1) the increment "dt" is going forward in an counterclockwise direction.

In the interior BVP, the boundary traction $Q(t) = \sigma_N(t) + i\sigma_{NT}(t)$ should satisfy the following equilibrium condition

$$\int_\Gamma \quad Qdt = 0,\ Re\int_\Gamma \quad Q\bar{t}dt = 0 \tag{5}$$

It is known that if the condition shown by Eq. (5) is satisfied, the BIE shown by Eq. (1) has many solutions. However, the existing solutions are not unique in general and they are different each other by the rigid body motions.

In the meantime, we define three rigid body displacement modes along the boundary as follows

$$U_{(1)}(t) = 1, \text{(or } u\text{=1, } v\text{=0, for } t \in \Gamma) \tag{6a}$$

$$U_{(2)}(t) = i, \text{(or } u\text{=0, } v\text{=1, for } t \in \Gamma) \tag{6b}$$

$$U_{(3)}(t) = it, \text{(or } u = -y_t,\ v = x_t, \text{ with } t = x_t + iy_t, \text{ for } t \in \Gamma) \tag{6c}$$

Clearly, $U_{(\gamma)}(t)$ (γ =1,2,3) represent three rigid motions for the boundary $t \in \Gamma$. It has been shown that $U(t) = U_{(\gamma)}(t)$ (γ =1,2,3) and Q(t)=0 is a solution of BIE shown by Eq. (1) (Chen and Wang 2015).

Generally, if one uses the BIE shown by Eq. (1) to an exterior boundary value problem, the increment "dt" should be going forward in a clockwise direction (Fig.1(b)). However, it is preferable to define increment "dt" in the anti-clockwise direction. In the case for the exterior boundary value problem (Fig. 1(b)), from Eq. (1) the relevant BIE for the exterior problem should be written as

$$\begin{aligned}\frac{U(t_o)}{2} - B_1 i \int_\Gamma \left(\frac{\kappa-1}{t-t_o}U(t)dt - L_1(t,t_o)U(t)dt + L_2(t,t_o)\overline{U(t)}dt\right) \\ = -B_2 i \int_\Gamma \left(2\kappa\, ln|t-t_o|\, Q(t)dt + \frac{t-t_o}{\bar{t}-\bar{t}_o}\overline{Q(t)}d\bar{t}\right), \quad (t_o \in \Gamma)\end{aligned} \tag{7}$$

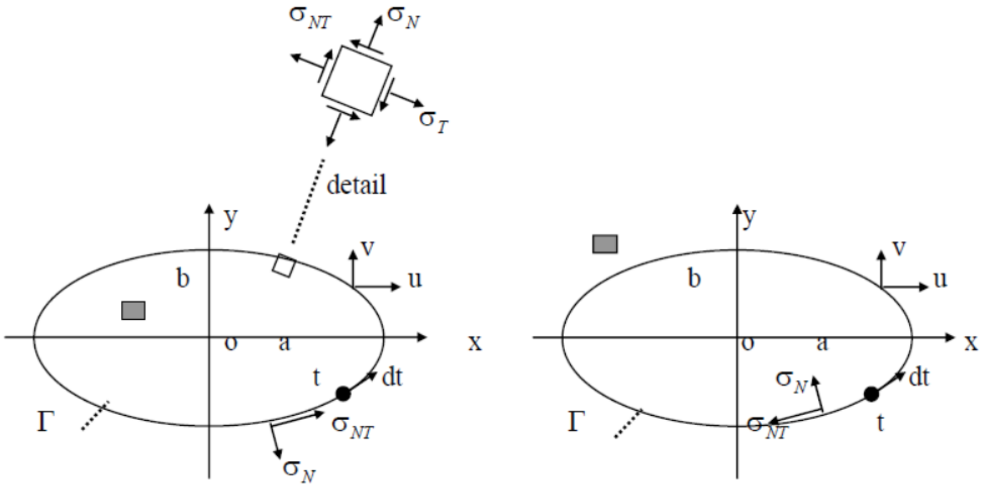

Fig. 1 (a) Interior boundary value problem, (b) Exterior boundary value problem, ▭ region define d

Note that in the exterior BVP, there is no constraint condition for the traction Q(t). In addition, for the arbitrarily assumed traction Q(t), there is a unique solution for displacement U(t).

2.2 Derivation for removal of rigid body motion in interior problem under the Neumann boundary value condition

In the formulation, U_1 and Q_1 denote the displacement and traction components applied along the boundary of the interior region (Fig. 1(a)). In addition, U_2 and Q_2 denote the displacement and traction components applied along the boundary of the exterior region (Fig. 1(b)). We assume the vector Q_1 used in the interior BVP and the vector Q_2 used in the exterior BVP take the same value, or $Q_1 = Q_2 = Q$. Cleary, the vector Q should satisfy the equilibrium conditions shown by Eq. (5).

After using the assumed conditions in Eqs. (1) and (7), we will obtain the following boundary integral equations

$$\frac{U_1(t_o)}{2} + B_1 i \int_\Gamma \left(\frac{\kappa - 1}{t - t_o} U_1(t)dt - L_1(t,t_o)U_1(t)dt + L_2(t,t_o)\overline{U_1(t)}dt \right)$$
$$= B_2 i \int_\Gamma \left(2\kappa \, ln|t - t_o| \, Q(t)dt + \frac{t-t_o}{\bar{t}-\bar{t}_o} \overline{Q(t)} d\bar{t} \right), (t_o \in \Gamma, \text{ for interior problem}) \tag{8}$$

$$\frac{U_2(t_o)}{2} - B_1 i \int_\Gamma \left(\frac{\kappa - 1}{t - t_o} U_2(t)dt - L_1(t,t_o)U_2(t)dt + L_2(t,t_o)\overline{U_2(t)}dt \right)$$
$$= -B_2 i \int_\Gamma \left(2\kappa \, ln|t - t_o| \, Q(t)dt + \frac{t-t_o}{\bar{t}-\bar{t}_o} \overline{Q(t)} d\bar{t} \right), (t_o \in \Gamma, \quad \text{for exterior problem}) \tag{9}$$

After taking the following steps: (a) adding Eq. (9) to (8), (b) subtracting Eq. (9) from (8), we will find the following coupled boundary integral equations

$$\frac{U_a(t_o)}{2} + B_1 i \int_\Gamma \left(\frac{\kappa-1}{t-t_o} U_b(t)dt - L_1(t,t_o)U_b(t)dt + L_2(t,t_o)\overline{U_b(t)}dt \right) = 0, (t_o \in \Gamma) \tag{10}$$

$$\frac{U_b(t_o)}{2} + B_1 i \int_{\Gamma} \left(\frac{\kappa - 1}{t - t_o} U_a(t)dt - L_1(t,t_o)U_a(t)dt + L_2(t,t_o)\overline{U_a(t)}dt\right)$$
$$= 2B_2 i \int_{\Gamma} \left(2\kappa\, ln|t - t_o|\, Q(t)dt + \frac{t-t_o}{\bar{t}-\bar{t}_o}\overline{Q(t)}d\bar{t}\right), (t_o \in \Gamma) \tag{11}$$

After making discretization to the coupled BIEs shown by Eqs. (10) and (11), we will obtain the following algebraic equations

$$\tfrac{1}{2}\{U_a\}+[H_U]\{U_b\} = \{0\} \tag{12}$$

$$\tfrac{1}{2}\{U_b\}+[H_U]\{U_a\} = [G_Q]\{Q\} \tag{13}$$

where two vectors are defined by

$$\{U_a\}=(\{U_1\} + \{U_2\}),\ \{U_b\}=(\{U_1\} - \{U_2\}) \tag{14}$$

Note that, the influence matrix $[H_U]$ in obtained after discretization of the integral operator $B_1 i \int_{\Gamma} \left(\frac{\kappa-1}{t-t_o} U(t)dt - L_1(t,t_o)U(t)dt + L_2(t,t_o)\overline{U(t)}dt\right)$, and the influence matrix $[G_Q]$ in obtained after discretization of the integral operator $B_2 i \int_{\Gamma} \left(2\kappa\, ln|t - t_o|\, Q(t)dt + \frac{t-t_o}{\bar{t}-\bar{t}_o}\overline{Q(t)}d\bar{t}\right)$.

From Eq. (12), we have

$$\{U_b\} = -\frac{1}{2}[H_U^{-1}]\{U_a\} \tag{15}$$

Substituting Eq. (15) into (13) yields

$$[H_{UC}]\{U_a\} = [G_Q]\{Q\} \tag{16}$$

where

$$[H_{UC}] =[H_U] - \tfrac{1}{4}[H_U^{-1}] \tag{17}$$

From Eq. (16), we have

$$\{U_a\}=[H_{UC}^{-1}][G_Q]\{Q\} \tag{18}$$

In addition, from and Eqs. (15) and (18), we can find the vector $\{U_b\}$. Further, from Eq. (14) we can find two vectors $\{U_1\}$ and $\{U_2\}$as follows

$$\{U_1\}=\{U_a\} + \{U_b\},\ \{U_2\}=\{U_a\} - \{U_b\} \tag{19}$$

Therefore, the solution for Neumann BVP for the interior region is obtained.

For examining the validity of the suggested method, we must provide an exact solution beforehand. In the exact solution, the displacement and traction vector assumed along the boundary Γ in the interior BVP are denoted by $\{U_1\}_{ext}$,$\{Q\}_{ext}$and $\{\sigma_T\}_{ext}$, respectively (Figs. 1 and 2).

After taking the following steps: (a) substituting $\{Q\}=\{Q\}_{ext}$ into right hand side of Eq. (13), and (b) completing the computation from Eqs. (14) to (19), we will obtain a unique solution $\{U_1\}_{num}$.

It is known that the relevant solution for the displacement vectors may be different each other by three rigid body motions. Therefore, it is inappropriate by comparing the obtained vector

$\{U_1\}_{num}$ with the vector $\{U_1\}_{ext}$ directly.

In fact, in the plane strain case, the strain component ε_T can be expressed from the Hook's law as follows (Fig. 1(a))

$$\varepsilon_T = \frac{(1-\nu^2)\sigma_T - \nu(1+\nu)\sigma_N}{2G(1+\nu)} \tag{20a}$$

Alternatively, we can express this relation in the form

$$\sigma_T = \frac{2G(1+\nu)\varepsilon_T + \nu(1+\nu)\sigma_N}{1-\nu^2} \tag{20b}$$

In Eq. (20), the component σ_N is from input data $\{Q\}_{ext}$, and ε_T is the strain in the T-direction which can be evaluated from the numerical solution of displacements along the boundary, or from $\{U_1\}_{num}$. Thus, the values of σ_T at many discrete points along the boundary Γ can be evaluated. The stress component vector σ_T evaluated from Eq. (20) is denoted by $\{\sigma_T\}_{num}$. After comparing this vector $\{\sigma_T\}_{num}$ with the vector $\{\sigma_T\}_{ext}$which is derived from the input data, we can judge the achieved accuracy accordingly.

It is well known that the equilibrium condition must be satisfied along the boundary in the interior boundary value problem. In addition, the displacement in the solution may not be unique. In order to obtain a unique solution in the interior boundary value problem, one needs to remove three kinds of rigid body motions (Blazquez *et al*. 1996)

In this paper, by solving the interior and exterior BVPs simultaneously, we need not to use the technique for removing three kinds of rigid body motions in the usual formulation.

3. Numerical example

One numerical example is provided to prove the efficiency of the suggested method. The plane strain condition and $\nu = 0.3$ are assumed. In the example, we provide a numerical solution for the Neumann BVP for a finite elliptic plate under the following conditions (Fig. 2)

$$\sigma_x = 0,\ \sigma_y = p,\ \sigma_{xy} = 0 \tag{21}$$

At the points along the boundary Γ, or $x = a\cos\theta$,$y = b\sin\theta$, the boundary tractions from the exact solution can be expressed as

$$\begin{gathered}\sigma_N = p\sin^2\beta,\quad \sigma_T = p\cos^2\beta,\ \sigma_{NT} = p\sin\beta\cos\beta,\\ \text{(with } \cos\beta = \frac{b\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}},\ \sin\beta = \frac{a\sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\text{)}\end{gathered} \tag{22}$$

In addition, it is easy to obtain the relevant solution for displacement

$$2G(u+iv) = -\nu px + (1-\nu)py,\ \text{(at point: } x = a\cos\theta,\ y = b\sin\theta\text{)} \tag{23}$$

After solving the Neumann BVP for interior region by solving interior and exterior BVPs simultaneously, the computed results for σ_T are expressed as

$$\sigma_T = f_1(\theta)p\ ,\ \text{(at the point } x = a\cos\theta,\ y = b\sin\theta\text{)} \tag{24}$$

For the following cases: b/a=0.25 and 0.75, the computed results of $f_1(\theta)$ (=σ_T/p) are plotted in Fig. 3. The exact solution is also plotted in the figure. We see from Fig. 3 that the numerical results are very accurate.

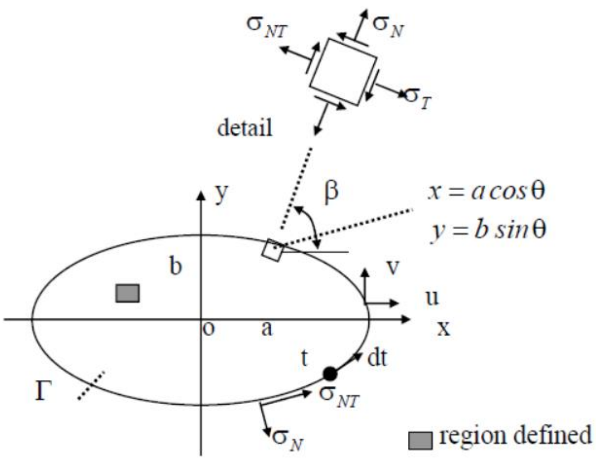

Fig. 2 Interior boundary value problem under the stress state (a) $\sigma_x = 0,\ \sigma_y = p,\ \sigma_{xy} = 0$ or (b) $\sigma_x = 0,\ \sigma_y = 0,\ \sigma_{xy} = q$

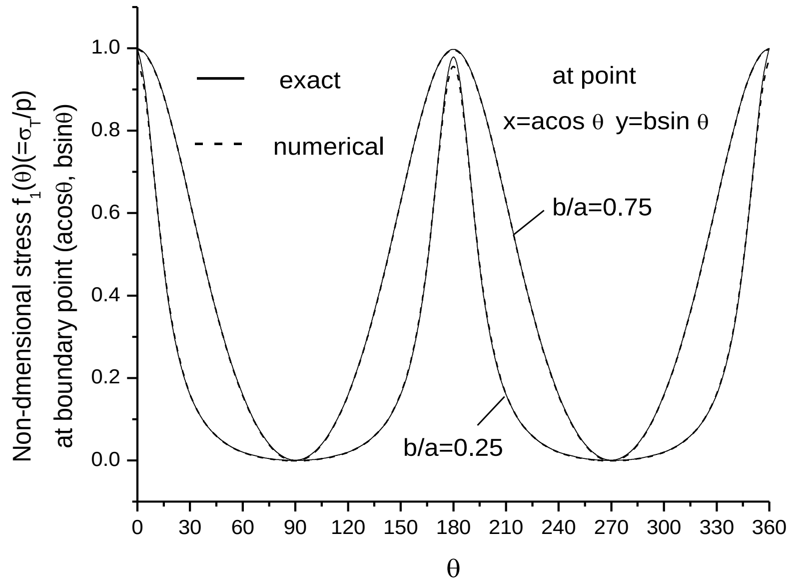

Fig. 3 Non-dimensional stress $f_1(\theta)$ (=σ_T/p) in the Neumann BVP under the conditions: (1) b/a=0.25 or b/a=0.75, (2) $\sigma_x = 0,\ \sigma_y = p,\ \ \sigma_{xy} = 0$ (see Eq. (24), Figs. 1 and 2)

4. Conclusions

In this paper, the properties of relevant integral operators are modified. The modified integral operators are generally invertible. Finally, a unique solution for boundary displacement of interior region can be obtained without using the technique for removal of rigid body motion.

In our formulation, the interior boundary value problem and the exterior boundary value problem are solved simultaneously. Therefore, the boundary traction $Q(t) = \sigma_N(t) + i\sigma_{NT}(t)$ should satisfy the equilibrium condition. This is an inconvenient point in the formulation.

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