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# Enumeration of axial rotation

Yong-San Yoon*

*Mechanical Engineering, Korea Advanced Institute of Science & Technology,*
*Yuseong-ku, Daejeon 305-701, Korea*

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**Abstract.** In this paper, two procedures of enumerating the axial rotation are proposed using the unit sphere of the spherical rotation coordinate system specifying 3D rotation. If the trajectory of the movement is known, the integration of the axial component of the angular velocity plus the geometric effect equal to the enclosed area subtended by the geodesic path on the surface of the unit sphere. If the postures of the initial and final positions are known, the axial rotation is determined by the angular difference from the parallel transport along the geodesic path. The path dependency of the axial rotation of the three dimensional rigid body motion is due to the geometric effect corresponding to the closed loop discontinuity. Firstly, the closed loop discontinuity is examined for the infinitesimal region. The general closed loop discontinuity can be evaluated by the summation of those discontinuities of the infinitesimal regions forming the whole loop. This general loop discontinuity is equal to the surface area enclosed by the closed loop on the surface of the unit sphere. Using this quantification of the closed loop discontinuity of the axial rotation, the geometric effect is determined in enumerating the axial rotation. As an example, the axial rotation of the arm by the Codman's movement is evaluated, which other methods of enumerating the axial rotations failed.

**Keywords:** axial rotation; geometric phase; prime geodesic; Codman's paradox; euler angle

## 1. Introduction

Even though we frequently use the angular rotations to describe the position of a three dimensional object in space, it is not clearly understood especially regarding to the axial rotation. The axial rotation is dependent on the trajectory (Miyazaki and Ishida 1991). There are numerous articles published about the axial rotations. Chao (1980) proposed a tri-axial goniometer to measure the joint angles by matching the yaw, pitch, and roll angles to the flexion-extension, abduction- adduction, and axial rotation angles respectively, which is used to define a joint coordinate system by Grood and Suntay (1983). However, this definition of axial rotation by Chao represents the pseudo-axial rotation different from the true axial rotation (Ishida 1990, Miyazaki and Ishida 1991, Crawford *et al*. 1999, Cheng *et al*. 2000). However, this pseudo-axial rotation is frequently denoted as the axial rotation even nowadays (Dennis *et al*. 2004, Digennaro *et al*. 2014, Fujimori *et al*. 2014). Another problem is that still many researchers were confused with the axial rotation resulting in some critical mistakes. For example, Miyazaki and Ishida (1991) proposed a method to calculate the axial rotation by integration of the angular velocity but missed to include

*Corresponding author, Professor Emeritus, E-mail: ysyoon@kaist.edu


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the geometric effect which contributes to the path dependency like the following two papers. Novotni *et al.* (2001) suggested a method to obtain the internal-external rotation. But their method is to produce the axial rotation not the internal-external rotation. Masuda *et al.* (2008) proposed a simple method of calculating the axial rotation from the Euler angles but the method is valid only when the motion is along the latitude line of the spherical rotation coordinate. Another problem is there are confusions when we are talking about "the rotation about the longitudinal axis" like at the Codman's movements. Cheng (2006) interpreted it as the roll-angle fixed while others (Miyazaki and Ishida 1991, Kawano *et al*. 2013) assumed the phrase as the axial rotation frozen.

The Codman's paradox since its publication (Codman 1934) about the shoulder rotation became popular topics to many researchers (Miyazaki and Ishida 1991, Politti *et al*. 1998, Stepan and Otahal 2006, Cheng 2008, Wolf *et al*. 2009, Mallon 2011, Kawano *et al*. 2013). But, this problem should not be limited to the shoulder joint. We may be easily ignorant of the same problem at other joints mainly because their motions are mostly limited to show some noticeable consequence. Politti *et al.* (1998) as well as Stepan and Otahal (2006) tried to identify Codman's paradox by the transformation matrices. Similarly, Mallon (2011) showed using the group theory that certain group of movements will produce same final position and described Codman's movement as those equivalent movements. The paradox could be viewed as a discontinuity in the axial rotation in a closed loop path movement with no axial rotation (Kawano *et al*. 2013). Cheng (2008) denoted this discontinuity of axial rotation in the Codman's movement as the equivalent axial rotation and proposed a hypothesis that this discontinuity is equal to the angle of swing in the movement. However, Cheng's hypothesis is valid only when the Codman's movements are interpreted as the ones with freezing the roll instead of the axial rotation.

Here, this paper is proposing two methods of enumerating the axial rotation: If the trajectory of the movement is known on the sphere of the spherical rotation coordinate system, the integration of the axial component of the angular velocity plus the geometric effect equal to the enclosed area subtended by the geodesic path on the surface of the unit sphere provides the axial rotation. If the postures of the initial and final positions are known, the axial rotation is obtained by the angular difference from the parallel transport along the geodesic path. Then, as an example, the Codman's movement is taken to demonstrate the enumeration of the axial rotation of the arm, which was not feasible by any other method of enumerating the axial rotation.

## 2. Angular velocity of axial rotation

A three dimensional angular position may be represented either by the quaternion, screw-axis rotation or Euler angles represented by the yaw, pitch, and roll of the gimbal mechanism as shown in Fig. 1. In a more intuitive way, we may use the axial rotation combined with its axis orientation which can be specified either by the projection angles or the position on the unit spherical surface denoted by the longitude and latitude. The last method is adopted in this study: the axial rotation plus longitude and latitude of the spherical rotation coordinate (Crawford *et al*. 1999, Cheng 2000). The axial rotation is calculated as the angular difference from the parallel transport along the geodesic path OA as shown in Fig. 2 when the initial and final postures are known. Otherwise, the axial rotation is obtained by integrating the axial component of the angular velocity plus the geometric effect equal to the enclosed area subtended by the geodesic path on the surface of the sphere.

Chao (1980) proposed the gyroscopic or three-axis Eulerian angle system to describe the three

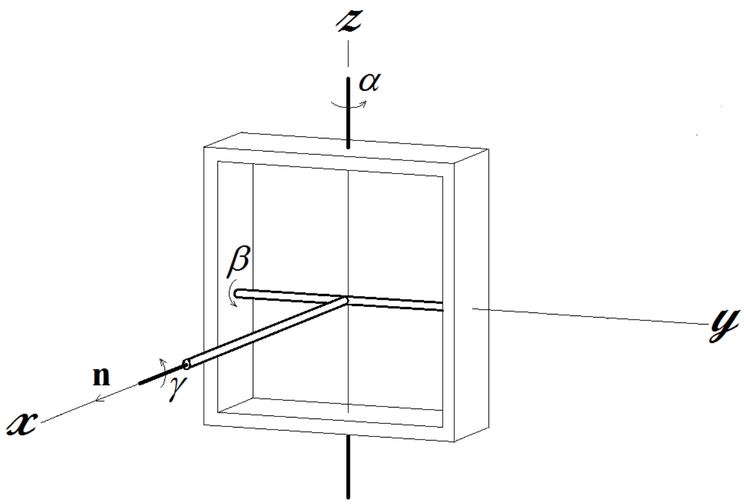


Fig. 1 Three axes gimbal mechanism with gyroscopic Euler angles

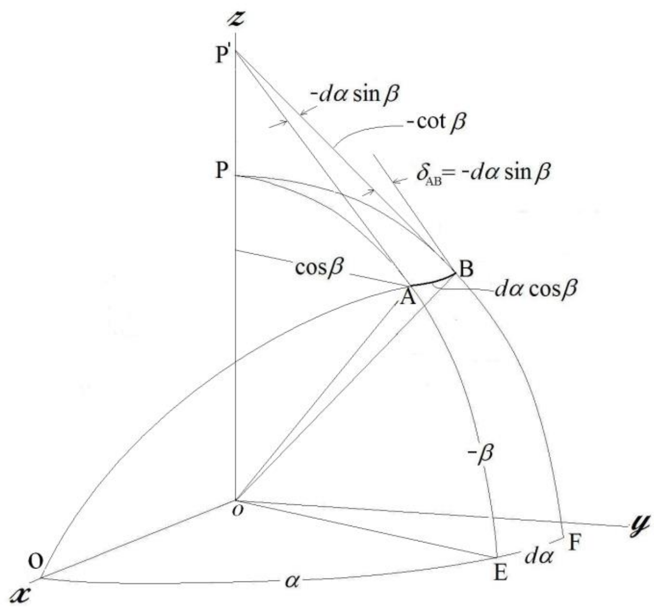


Fig. 2 Infinitesimal movement on geographical sphere

dimensional movement of a joint since it is more convenient to use by matching with the clinical definition of joint motion at the neutral position. The yaw, pitch and roll angles were defined as the flexion-extension, abduction-adduction, and axial rotation angles of the joint respectively. However, this definition of the axial rotation represents the pseudo-axial rotation (Miyazaki and Ishida, 1991) different from the true axial rotation. Miyazaki and Ishida suggested obtaining the axial rotation by the integration of the axial component of the angular velocity along the path from the original position to the final position. The axial component of the angular velocity is

$$\omega_a = \dot{\gamma} + \dot{\alpha} \cdot \sin\beta \tag{1}$$

However, such integration procedure may provide an incorrect result as it missed to include the geometric effect equal to the enclosed area subtended by the geodesic path on the surface of the sphere. This will be discussed in detail in the following section.

## 3. Closed loop discontinuity of axial rotation

In this section, the closed loop discontinuity of the axial rotation will be demonstrated by showing the axial rotation after parallel transporting (no axial rotation) along a closed loop on the surface of the spherical rotation coordinate system. For simplicity, let's limit our movements in this section as the parallel transport along the longitude and latitude lines in the spherical rotation coordinate system. Then, for the infinitesimal movement from A to B, the azimuth representing the offset from the North Pole is changed as shown in Fig. 2 by the amount corresponding to the second term of Eq. (1)

$$\delta_{AB} = -d\alpha \sin\beta \tag{2}$$

Similarly, if the movement is made from C to D as shown in Figure 3, then the azimuth change $\delta_{CD}$ becomes

$$\delta_{CD} = -d\alpha \sin(\beta + d\beta) \tag{3}$$

If A and B are on the Equator, $\beta$=0 and the azimuth change $\delta_{AB}$ becomes

$$\delta_{AB} = 0 \tag{4}$$

As another extreme case, consider when $\beta$=90$^o$ with C and D on the North Pole. We have to note that the pole is a singular point and we should avoid the point conceptually by replacing the

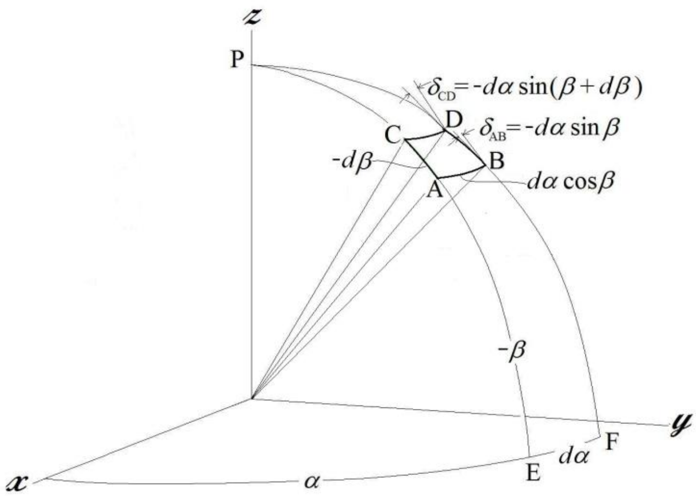


Fig. 3 Discontinuity in axial rotation on closed loop

pole with a circular path with infinitesimal radius around the pole. Then, the azimuth change $\delta_{CD}$ becomes

$$\delta_{CD} = -d\alpha \tag{5}$$

Thus, these effects are path dependent and contribute to a discontinuity when a path is making a closed loop. For example, the $\omega$-zero movement along the closed loop (A, B, D, C, A) forming an infinitesimal rectangle yields the closed loop discontinuity in the axial rotation

$$d\delta = \delta_{AB} - \delta_{CD} = d\alpha\,(\sin(\beta + d\beta) - \sin\beta) = d\alpha\; d\beta\; \cos\beta \tag{6}$$

as the path along the longitudinal lines does not contribute to the azimuth change. This closed loop discontinuity is equal to the surface area of the rectangle ABDC as shown in Fig. 3. For any closed loop, the region can be subdivided into those infinitesimal rectangles. As the discontinuity of each rectangle can be summed up, the total discontinuity of the closed loop should be equal to the enclosed area matching to the Kelvin-Tait theorem (Kelvin and Tait 1912). This discontinuity is called as the geometric phase (Berry 1990) or Berry's phase.

## 4. Enumeration of axial rotation

Now, two procedures of enumerating the axial rotation are proposed using the unit sphere of the geographic coordinate system specifying 3D rotation. In the first procedure when the trajectory of the movement is known, the integration of the axial component of the angular velocity plus the geometric effect equal to the enclosed area subtended by the geodesic path on the surface of the unit sphere provide the axial rotation

$$\theta = \int \omega_a \, dt + \Delta\delta \tag{7}$$

where $\Delta\delta$ is the geometric effect equal to the surface area enclosed by the trajectory curve and the geodesic path as shown in Fig. 4.

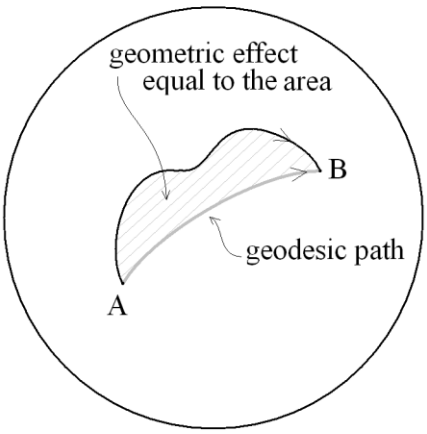


Fig. 4 Geometric effect in enumerating axial rotation

When the destination and the starting positions are exactly at the opposite of the sphere, the geodesic line can not be determined uniquely at a joint. In that case, we should adopt the central direction of the range of motion at the joint and the prime geodesic path is determined by the plane encompassing that direction. For example, at the shoulder joint, the sagittal plane should be taken to determine the prime geodesic path for the motion from the south pole to the north pole as it encompasses the central direction of the range of motion at the shoulder joint.

In the second procedure when the postures of the initial and final position are known, the axial rotation is obtained by the angular difference from the parallel transport along the geodesic path. This procedure would produce the same axial rotation as the two-step method (Cheng *et al*.2000), and the tilt and twist method (Crawford *et al*. 1999). But all those methods would fail if two positions are exactly opposite on the sphere.

## 5. Application to Codman's movements

Now, let's deal with Codman's paradox which notes the longitudinal axis rotation after two or three sequential arm rotations that involve no rotation about the longitudinal axis. The closed loop motions by those sequential rotations are named as Codman's rotations.

There are two kinds of original Codman's movement: 90° movement and 180° movement. The 90° movement consists of the South Pole, and two points at the Equator making a closed loop while the 180° movement consists of the South and North Poles, and two points at the Equator making a closed loop. Fig. 5 shows the axial rotations corresponding to the 90° movement. The first path is the direct movement from the point C to the point D along the equatorial path. The second path is also the movement from the point C to the point D but via the point A on the way to the point D. Thus the second path produces the geometric effect of $\pi/2$ while the first path produces none as can be shown in Fig. 5.

Fig. 6(a) shows the 180° movement passing through the points A, C, E, D, B, and A. Here, the points A, E, and B are at the Poles and the points C and D are at the Equator. In this closed loop (A-C-E-D-B-A), the magnitude of discontinuity of the axial rotation is equal to the surface area $\pi$

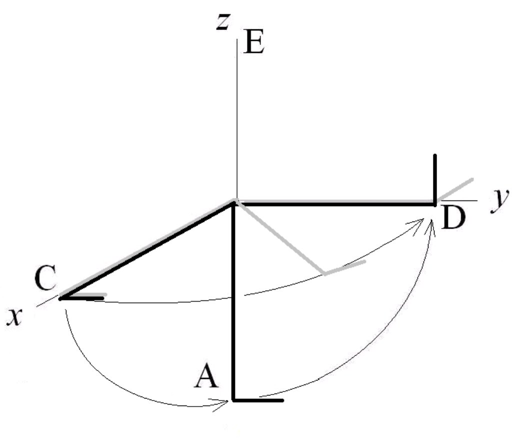


Fig. 5 Codman's 90° movement and axial rotation

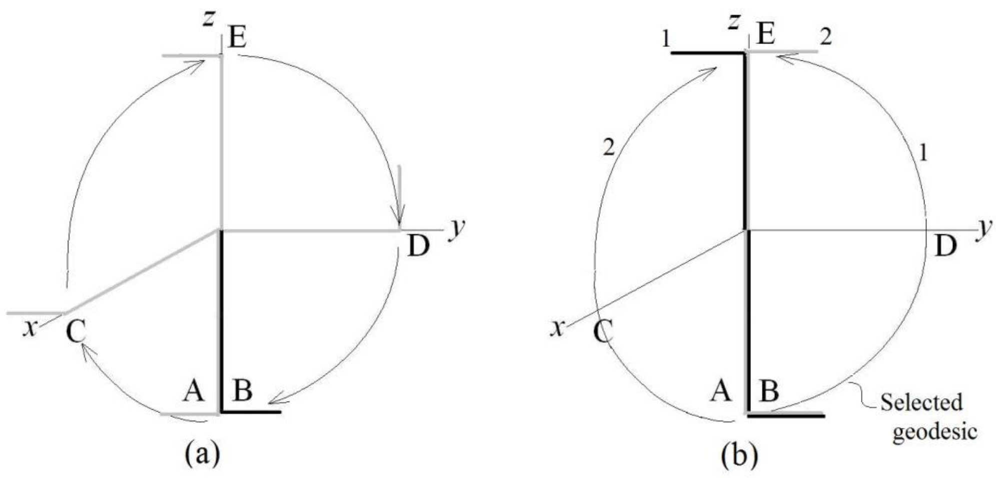


Fig. 6 Codman's 180° movement and axial rotation

of the region ACEDBA as we are observing in Codman's 180° movement (Kawano *et al*. 2013).

Now, for this movement we may enumerate the axial rotations for the motions A-C-E and B-D-E as shown in Fig. 5(b). As the point A and point E are at exactly opposite position of the sphere, we may choose the x-plane (sagittal plane) in constructing the prime geodesic path as explained in the previous section. Then the path B-D-E is the geodesic one and the geometric effect is zero while the path A-C-E has geometric effect of $\pi$ with the prime geodesic path. Thus those two movements produced different axial rotations due to the different geometric effects yielding the difference of $\pi$. As we know the starting and final positions, we may also apply the second method to see that the second path generates the axial rotation of $\pi$. Other published methods (Miyazaki and Ishida 1991, Crawford *et al*. 1999, Cheng *et al*. 2000, Novotony *et al*. 2001, Masuda *et al*. 2008) do not work for these examples.

## 6. Discussions

There are numerous articles published about the axial rotations. Chao (1980) proposed a triaxial goniometer to measure the joint angles by matching yaw, pitch, and roll angles to flexion-extension, abduction- adduction, and axial rotation angles respectively. However, the axial rotation defined this way produces pseudo-axial rotation (Miyazaki and Ishida 1991) instead of the true axial rotation. Many researchers proposed methods to calculate the true axial rotations. When the initial and final postures are given, Cheng's two-step method (2000) may work. Otherwise, all those endeavors to get the axial rotation by integrating the axial component of the angular velocity failed as they missed the geographic effect to supplement (Miyazaki and Ishida 1991, Novotony *et al*. 2001, Masuda *et al*. 2008).

The Codman's paradox also became mind boggling to many researchers after its publication (Codman 1934) as the contribution of the geometric phase (Kawano *et al*. 2013) was not well publicized. Politti *et al.* (1998) showed that three successive rotations in the Codman's 90° movement correspond to a single 90° rotation around its axis using the multiplication of the rotational matrices, which is also confirmed by Vladimir and Otahal (2006). Similar endeavor was made by Mallon (2011) using the group theory that certain group of motion will produce same final position. However, that group should not be exclusive as any motion having same starting and end points on the unit spherical surface will produce same final position as long as its axial rotation is properly adjusted initially. That amount of the axial rotation to be adjusted is equal to the discontinuity in the axial rotation along a closed loop. Cheng (2008) denoted this discontinuity of axial rotation in the Codman's movement as the equivalent axial rotation and proposed a hypothesis that this discontinuity is equal to the angle of swing in the first movement of 90° and twice the angle of swing in the second movement of 180°. He showed the hypothesis was working for the several examples he chosen. But Cheng's procedure might include some error as his enumeration method of the axial rotation does not work for those problems.

The Codman's 180° movement possesses two singular poles and every methods of enumerating the axial rotation quoted in this paper failed as they didn't consider the geometric effect. To handle this problem, the prime geodesic is introduced here based on the central direction of the range of motion at a joint.

## 7. Conclusions

In this paper, two procedures of enumerating the axial rotation are proposed using the unit sphere of the geographic coordinate system specifying 3D rotation. If the trajectory of the movement is known, the integration of the axial component of the angular velocity plus the geometric effect equal to the enclosed area subtended by the geodesic path on the spherical surface. If the postures of the initial and final position are known, the axial rotation is obtained by the angular difference from the parallel transport along the geodesic path. If two points are exactly opposite on the spherical surface in taking the geodesic path, some preference like the central direction of the range of motion at a joint might be considered to determine the prime geodesic path.

As an example of enumerating the axial rotations, the Codman's movement which possesses several singular points is handled successfully, for which other methods of enumerating the axial rotations failed to work.

## References

Berry, M.V. (1990), "Anticipations of the Geometric Phase", *Phys. Today*, **43**(12), 34-40.

Chao, E.Y. (1980), "Justification of triaxial goniometer for the measurement of joint rotation", *J. Biomech.*, **13**(12), 989-1006.

Cheng, P.L., Nicol, A.C. and Paul, J.P. (2000), "Determination of axial rotation angles of limb segments - a new method", *J. Biomech.*, **33**(7), 837-843.

Cheng, P.L. (2000), "A spherical rotation coordinate system for the description of three-dimensional joint rotations", *Ann. Biomed. Eng.*, **28**(11), 1381-92.

Cheng, P.L. (2006), "Simulation of Codman's paradox reveals a general law of motion", *J. Biomech.*, **39**(7), 1201-1207.

Codman, E.A. (1934), *The shoulder: Rupture of the supraspinatus tendon and other lession in or about the subacromial bursa*, 2nd Edition, T. Todd Co., Boston.

Crawford, N.R., Yamaguchi, G.T. and Dickman, C.A. (1999), "A new technique for determining 3-D joint angles: the tilt/twist method", *Clin. Biomech*., **14**, 153-165.

Dennis, D.A., Komistek, R.D., Mahfouz, M.R., Walker, S.A. and Tucker, A. (2004), "A multicenter analysis of axial femorotibial rotation after total knee arthroplasty", *Clin. Orthop. Relat. Res*., **428**, 180-9.

Digennaro, V., Zambianchi, F., Marcovigi, A., Mugnai, R., Fiacchi, F. and Catani, F. (2014), "Design and kinematics in total knee arthroplasty", *Int. Orthop*., **38**, 227-233.

Grood, E.S. and Suntay, W.J. (1983), "A joint coordinate system for the clinical description of three-dimensional motions: application to the knee", *J. Biomech. Eng*., **105**(2), 136-144.

Ishida, A. (1990), "Definition of axial rotation of anatomical joints", *Front. Med. Biol. Eng*., **2**(1), 65-68.

Kawano, D.T., Novelia, A. and O'Reilly, O.M. (2013), "Codman's paradox", Rotations, Lecture Note, http://rotations.berkeley.edu/?page_id=1208.

Kelvin, L. and Tait, P.G. (1912), *A Treatise on Natural Philosophy*, Part 1, Reprinted 6th Edition, Cambridge University Press, Cambridge.

Mallon, W.J. (2011), "On the hypotheses that determine the definitions of glenohumeral joint motion: with resolution of Codman's pivotal paradox", *J. Shoulder. Elbow. Surg*., **21**, 1-16.

Masuda, T., Ishida, A., Cao, L. and Morita, S. (2008), "A proposal for a new definition of the axial rotation angle of the shoulder joint", *J. Electromyogr. Kinesiol*., **18**(1), 154-159.

Miyazaki, S. and Ishida, A. (1991), "New mathematical definition and calculation of axial rotation of anatomical joints", *J. Biomech. Eng*., **113**(3), 270-275.

Novotny, J.E., Beynnon, B.D. and Nichols, C.E. 3rd. (2001), "A numerical solution to calculate internal-external rotation at the glenohumeral joint", *Clin. Biomech. (Bristol, Avon)*, **16**(5), 395-400.

Politti, J.C., Goroso, G., Valentinuzzi, M.E. and Bravo, O. (1998), "Codman's paradox of the arm rotations is not a paradox: mathematical validation", *Med. Eng. Phys*., **20**, 257-260.

Stepan, V. and Otahal, S. (2006), "Is Codman's paradox really a paradox?", *J. Biomech*., **39**(16), 3080-3082.

Wolf, S.I., Fradet, L. and Rettig, O. (2009), "Conjunct rotation: Codman's paradox revisited", *Med. Biol. Eng. Comput*., **47**(5), 551-556.

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