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An inverse hyperbolic theory for FG beams resting on Winkler-Pasternak elastic foundation

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Abstract. Bending, buckling and free vibration responses of functionally graded (FG) higher-order beams resting on two parameter (Winkler-Pasternak) elastic foundation are studied using a new inverse hyperbolic beam theory. The material properties of the beam are graded along the thickness direction according to the power-law distribution. In the present theory, the axial displacement accounts for an inverse hyperbolic distribution, and the transverse shear stress satisfies the traction-free boundary conditions on the top and bottom surfaces of the beams. Hamilton's principle is employed to derive the governing equations of motion. Navier type analytical solutions are obtained for the bending, bucking and vibration problems. Numerical results are obtained to investigate the effects of power-law index, length-to-thickness ratio and foundation parameter on the displacements, stresses, critical buckling loads and frequencies. Numerical results by using parabolic beam theory of Reddy and first-order beam theory of Timoshenko are specially generated for comparison of present results and found in excellent agreement with each other.

Keywords: inverse hyperbolic beam theory; FG beam; displacements; stresses; critical buckling load; frequencies, Winkler-Pasternak elastic foundation

1. Introduction

A functionally graded (FG) material is formed by varying the microstructure from one material to another material with a specific gradient. Now days, the application of FG material in engineering structures is growing rapidly due to their attractive properties (Koizumi 1993, 1997, Muller *et al*. 2003, Pompe *et al*. 2003, Schulz *et al*. 2003). Therefore, understanding bending, buckling and free vibration responses of FG beams becomes an important task. Few researchers have developed elasticity solutions for the analysis of FG beams (Sankar 2001, Zhong and Yu 2007, Ding *et al*. 2007, Ying *et al*. 2008, Daouadji *et al*. 2013) which are analytically very difficult. Therefore, researchers have developed various approximate beam theories which are mathematically simpler compared to elasticity solutions. The Euler-Bernaulli beam theory and the first-order beam theory (Timoshenko 1921) are not suitable for the analysis of thick beams due to neglect of shear deformation effect. Therefore, higher-order beam theories are developed by

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various researchers which considered the effect of transverse shear deformation and accurate for the analysis of thick beams. There exist various classes of higher-order beam theories which account the effect of transverse shear deformation such as parabolic beam theories (Reddy 1984, Sayyad *et al*. 2015a), trigonometric beam theories (Touratier 1991, Mantari *et al*. 2012a, b, Neves *et al*. 2012a, Sayyad and Ghugal 2011a, Sayyad *et al*. 2015b), hyperbolic beam theories (Soldatos 1992, Neves *et al*. 2012b, Sayyad and Ghugal 2011b), exponential beam theories (Karama *et al*. 2003), etc. Recently, Sayyad and Ghugal (2015, 2017a) presented a comprehensive literature review on various higher-order beam theories for the analysis of beam and plate structures. Several research papers have been published by researchers in last decade on bending, buckling and free vibration analysis of functionally graded beams (Thai and Vo 2012, Sayyad and Ghugal 2017b, Simsek 2010, Hadji *et al*. 2016a, b, Bourada *et al*. 2015, Vo *et al*. 2014a, b).

Fazzolari (2016) presented the free vibration analysis of three-dimensional metallic and functionally graded beams with arbitrary boundary conditions using refined variable-kinematics of quasi-3D beam theories hierarchically generated by using the method of power series expansion of displacement components. Ghumare and Sayyad (2017) have developed a new fifth-order shear and normal deformation theory for static bending and elastic buckling of functionally graded beams. Recently, Fazzolari (2018) investigated the free vibration and elastic stability behaviour of three-dimensional functionally graded sandwich beams featured by two different types of porosity, with arbitrary boundary conditions and resting on Winkler-Pasternak elastic foundations. This investigation is carried out by using exponential, polynomial, and trigonometric higher-order beam theories.

Hyperbolic function considering the effect of transverse shear deformation was first time suggested by Soldatos (1992). The theory is called as hyperbolic beam theory which is further used by the many researchers for the analysis of isotropic, laminated composite, sandwich and functionally graded beams and plates. Later on, researchers have suggested various hyperbolic functions which take into account shear deformation (Meiche *et al*. 2011, Akavci and Tanrikulu 2008, Mahi *et al*. 2015, Grover *et al*. 2013).

In this study, a new inverse hyperbolic function suggested by Nguyen *et al*. (2016) is used to develop an inverse hyperbolic beam theory. This theory is applied for the bending, buckling and free vibration analysis of functionally graded beams resting on two parameter elastic foundation. The material properties of functionally graded beams are varied through the thickness according to the power-law distribution. The theory gives hyperbolic cosine distribution of transverse shear stress through the thickness of the beam and satisfies the traction free boundary conditions on the top and bottom surfaces without using the problem dependent shear correction factor. Governing equations of motion are obtained using the Hamilton's principle. Analytical solutions for simply supported boundary condition are obtained using Navier's solution technique. Numerical results are compared with those reported previously in literature. The effects of the power-law index, length-to-thickness ratio and foundation parameter on the displacements, stresses, critical buckling loads and natural frequencies of FG beams are investigated.

2. Theoretical formulation

Consider a beam resting on two parameter elastic foundation as shown in Fig. 1 with length $\boldsymbol{L}$ and rectangular cross-section **($\boldsymbol{b}\times\boldsymbol{h}$).** The beam is made of functionally graded material such that the bottom surface of beam is ceramic rich and top surface is metal rich. The origin of the Cartesian coordinate system is assumed at the left support of the beam. The x-axis coincides

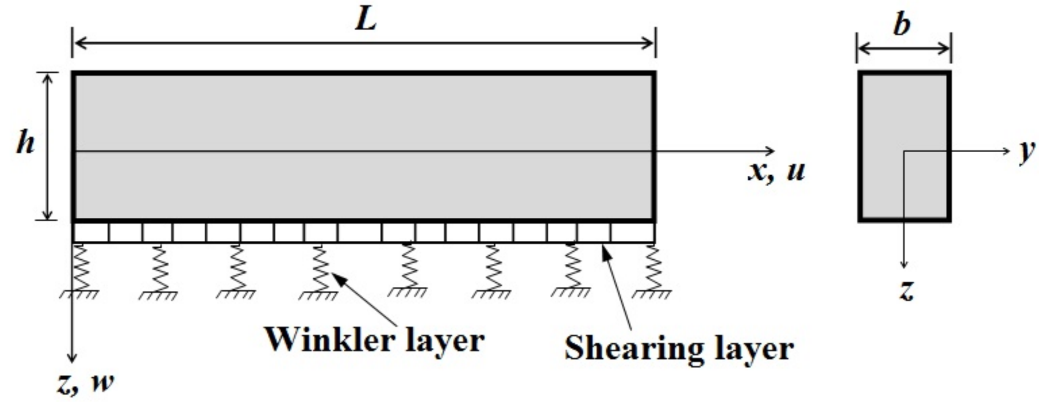

Fig. 1 FGM Beam resting on two parameter elastic foundation

with the beam neutral axis and the z-axis is assumed positive downward. The beam is assumed to be placed on the two-parameter elastic foundation, including Winkler layer and Pasternak shearing layer with stiffnesses k_w and k_p respectively. The relation between these stiffnesses and the transverse deflection (w) of the FG beam is given by

$$R = k_w w - k_p \frac{d^2 w}{dx^2}, \tag{1}$$

where $\boldsymbol{R}$ is the reaction force of foundation.

The functionally graded beam is made of ceramic (alumina) and metal (aluminum) materials. The material property distribution of functionally graded beams through the thickness is given by the power-law:

$$P(z) = P_c V_c + (1 - V_c) P_m$$
$$V_c = \left[\frac{1}{2} + \frac{z}{h}\right]^p \tag{2}$$

where P_c and P_m are the properties (Young's modulus $\boldsymbol{E}$**,** Poisson's ratio $\boldsymbol{\mu}$**,** shear modulus $\boldsymbol{G}$**,** and density $\boldsymbol{\rho}$) of ceramic and metal, respectively; $\boldsymbol{p}$ is the power-law index and V_c is the volume fraction. The variation of the volume fraction $\boldsymbol{V_c}$ through the thickness of the beam for various values of the power-law index is shown in Fig. 2.

2.1 An inverse hyperbolic beam theory (IHBT)

To include the effect of transverse shear deformation and rotary inertia, the present theory is developed based on the following kinematical assumptions: (1) the axial displacement consists of the extension, bending and shear components; (2) The axial displacement in x-direction accounts for an inverse hyperbolic distribution; (3) there is no relative motion in the y-direction at any points in the cross section of the beam; (4) the theory gives hyperbolic cosine distribution of transverse shear strain across the thickness of the beam and satisfies zero traction boundary conditions on the top and bottom surfaces of the beam. Based on these assumptions, the

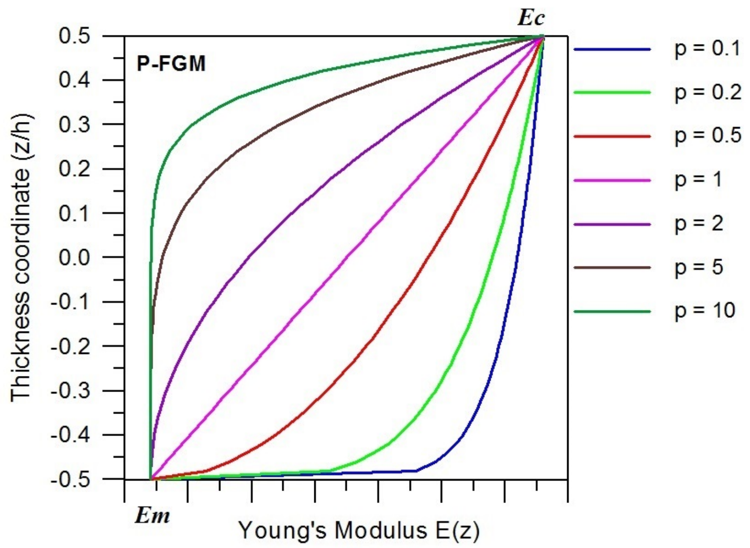

Fig. 2 Variation of Young's modulus $E(z)$ through the thickness of FG beam for various values of the power-law index (p)

displacement field of the present inverse hyperbolic shear deformation theory is given by:

$$\begin{aligned} u(x,z) &= u_0(x) - z\frac{dw_0(x)}{dx} + f(z)\phi(x) \\ w(x) &= w_0(x) \end{aligned} , \tag{3}$$

where

$$f(z) = \cot^{-1}\left(\frac{h}{z}\right) - \left(\frac{16z^3}{15h^3}\right), \tag{4}$$

where $\boldsymbol{u}$ and $\boldsymbol{w}$ are the displacements in x and z- directions respectively; $\boldsymbol{u_0}$ and $\boldsymbol{w_0}$ are displacements of a point on the neutral axis ($z = 0$) of the beam. An inverse hyperbolic shape function is assumed according to transverse shearing strain distribution across the thickness of the beam. The non-zero strains associated with the displacement field in Eq. (3) are:

$$\varepsilon_x = \varepsilon_x^0 + zk_x^b + f(z)\varepsilon_x^1, \quad \gamma_{zx} = g(z)\phi, \tag{5}$$

where

$$\begin{gathered} \varepsilon_x^0 = \frac{du_0}{dx}, \quad k_x^b = -\frac{d^2w_0}{dx^2}, \quad \varepsilon_x^1 = \frac{d\phi}{dx}, \\ f(z) = \left[\cot^{-1}\left(\frac{h}{z}\right) - \left(\frac{16z^3}{15h^3}\right)\right] \quad \text{and} \quad g(z) = \left[\frac{h}{h^2+z^2} - \frac{48z^2}{15h^3}\right], \end{gathered} \tag{6}$$

The stress-strain relationships at any point within the beam are given by one dimensional Hooke's laws as follows:

$$\sigma_x = E(z)\varepsilon_x \quad \text{and} \quad \tau_{zx} = G(z)\gamma_{zx}, \tag{7}$$

2.2 Governing equations of motion

The variational strain energy (δU) due to internal forces, variational potential energy (δV) due to external forces and stiffness of elastic foundation; and variational kinetic energy (δK) due to inertia forces are required to derive the governing equations of motion.

The variation of the strain energy (δU) can be stated as:

$$\delta U = b\int_0^L \int_{-h/2}^{h/2} \left(\sigma_x \delta\varepsilon_x + \tau_{zx}\delta\gamma_{zx}\right) dz\, dx = \int_0^L \left(N_x \frac{d\delta u_0}{dx} - M_x^c \frac{d^2\delta w_0}{dx^2} + M_x^s \frac{d\delta\phi}{dx} + Q_{zx}\delta\phi \right) dx, \tag{8}$$

where $N_x, M_x^c, M_x^s, Q_{zx}$ are the stress resultants in terms of the axial force, bending moment, higher order moment and shear force, respectively as defined below:

$$\begin{aligned} N_x &= b\int_{-h/2}^{h/2} \sigma_x\, dz = A_{11}\frac{du_0}{dx} - B_{11}\frac{d^2 w_0}{dx^2} + C_{11}\frac{d\phi}{dx} \\ M_x^c &= b\int_{-h/2}^{h/2} \sigma_x\, z\, dz = B_{11}\frac{du_0}{dx} - D_{11}\frac{d^2 w_0}{dx^2} + E_{11}\frac{d\phi}{dx} \\ M_x^s &= b\int_{-h/2}^{h/2} \sigma_x\, f(z)\, dz = C_{11}\frac{du_0}{dx} - E_{11}\frac{d^2 w_0}{dx^2} + F_{11}\frac{d\phi}{dx} \\ Q_{zx} &= b\int_{-h/2}^{h/2} \tau_{zx}\, g(z)\, dz = H_{55}\phi \end{aligned} \tag{9}$$

where

$$\begin{aligned} A_{11} &= b\int_{-h/2}^{h/2} E(z)\,dz, \quad B_{11} = b\int_{-h/2}^{h/2} E(z)\,z\,dz, \quad C_{11} = b\int_{-h/2}^{h/2} E(z) f(z)\,dz, \\ D_{11} &= b\int_{-h/2}^{h/2} E(z) z^2\,dz, \quad E_{11} = b\int_{-h/2}^{h/2} E(z)\,z\, f(z)\,dz, \\ F_{11} &= b\int_{-h/2}^{h/2} E(z)\left[f(z)\right]^2 dz, \quad H_{55} = b\int_{-h/2}^{h/2} G(z)\left[g(z)\right]^2 dz \end{aligned} \tag{10}$$

The variation of the potential energy (δV) due to external forces and stiffness of elastic foundation can be written as

$$\delta V = \int_0^L \left(q\,\delta w + N_0 \frac{dw}{dx}\frac{d\delta w}{dx} - R\,\delta w \right) dx, \tag{11}$$

The variation of kinetic energy (δK) can be written in following form,

$$\delta K = b\int_0^L \int_{-h/2}^{h/2} \rho(z)\left(\frac{d^2u}{dt^2}\delta u + \frac{d^2w}{dt^2}\delta w\right)dz\,dx$$
$$= \int_0^L \left(I_A\frac{d^2u}{dt^2} - I_B\frac{d^3w_0}{dxdt^2} + I_C\frac{d^2\phi}{dt^2}\right)\delta u_0 dx + \int_0^L \left(-I_B\frac{d^2u}{dt^2} + I_D\frac{d^3w_0}{dxdt^2} - I_E\frac{d^2\phi}{dt^2}\right)\frac{d\delta w_0}{dx}dx \tag{12}$$
$$+\int_0^L \left(I_C\frac{d^2u}{dt^2} - I_E\frac{d^3w_0}{dxdt^2} + I_F\frac{d^2\phi}{dt^2}\right)\delta\phi dx + \int_0^L \left(I_A\frac{d^2w_0}{dt^2}\right)\delta w_0 dx\ ,$$

where $\rho(z)$ is the mass density and I_A, I_B, I_C, I_D, I_E, I_F are the inertia coefficients as defined below:

$$I_A = b\int_{-h/2}^{h/2}\rho(z)dz,\quad I_B = b\int_{-h/2}^{h/2}\rho(z)z\,dz,\quad I_C = b\int_{-h/2}^{h/2}\rho(z)f(z)dz,$$
$$I_D = b\int_{-h/2}^{h/2}\rho(z)z^2\,dz,\quad I_E = b\int_{-h/2}^{h/2}\rho(z)z\,f(z)dz,\quad I_F = b\int_{-h/2}^{h/2}\rho(z)\left[f(z)\right]^2 dz\ , \tag{13}$$

Governing equations of motion of the present inverse hyperbolic shear deformation theory are derived using Hamilton's principle stated in Eq. (14),

$$\int_{t_1}^{t_2}\left(\delta U - \delta V + \delta K\right)dt = 0\ , \tag{14}$$

where t_1 and t_2 are the initial time and final time, respectively. Substituting Eqs. (8)-(13) into the Eq. (14) and integrating by parts and setting the coefficients of δu_0, δw_0 and $\delta\phi$ equal to zero, the following equations of motion are obtained,

$$\frac{dN_x}{dx} = I_A\frac{d^2u_0}{dt^2} - I_B\frac{d^3w_0}{dxdt^2} + I_C\frac{d^2\phi}{dt^2}$$
$$\frac{d^2M_x^c}{dx^2} = -q + k_w - k_p\frac{d^2w_0}{dx^2} + N_0\frac{d^2w_0}{dx^2} + I_B\frac{d^3u_0}{dxdt^2} - I_D\frac{d^4w_0}{dx^2dt^2} + I_A\frac{d^2w_0}{dt^2} + I_E\frac{d^3\phi}{dxdt^2} \tag{15}$$
$$\frac{dM_x^s}{dx} - Q_{zx} = I_C\frac{d^2u_0}{dt^2} - I_E\frac{d^3w_0}{dxdt^2} + I_F\frac{d^2\phi}{dt^2}\ ,$$

By substituting the stress resultants from Eq. (9) into Eq. (15), the following governing equations of motion can be obtained,

$$A_{11}\frac{d^2u_0}{dx^2} - B_{11}\frac{d^3w_0}{dx^3} + C_{11}\frac{d^2\phi}{dx^2} = I_A\frac{d^2u_0}{dt^2} - I_B\frac{d^3w_0}{dxdt^2} + I_C\frac{d^2\phi}{dt^2}$$
$$B_{11}\frac{d^3u_0}{dx^3} - D_{11}\frac{d^4w_0}{dx^4} + E_{11}\frac{d^3\phi}{dx^3} = -q + k_w - k_p\frac{d^2w_0}{dx^2} + N_0\frac{d^2w_0}{dx^2} + I_B\frac{d^3u_0}{dxdt^2}$$
$$-I_D\frac{d^4w_0}{dx^2dt^2} + I_A\frac{d^2w_0}{dt^2} + I_E\frac{d^3\phi}{dxdt^2} \tag{16}$$
$$C_{11}\frac{d^2u_0}{dx^2} - E_{11}\frac{d^3w_0}{dx^3} + F_{11}\frac{d^2\phi}{dx^2} - H_{55}\phi = I_C\frac{d^2u_0}{dt^2} - I_E\frac{d^3w_0}{dxdt^2} + I_F\frac{d^2\phi}{dt^2}\ ,$$

3. Analytical solutions

Navier-type analytical solutions are obtained for the bending, buckling and free vibration analysis of functionally graded beams resting on two parameter elastic foundation. According to the Navier-type solution technique, the unknown displacement variables and the transverse load are expanded in a Fourier series as given below:

$$u_0(x,t)=\sum_{m=1,3,5}^{\infty} u_m \cos\alpha x e^{i\omega t},\quad w_0(x,t)=\sum_{m=1,3,5}^{\infty} w_m \sin\alpha x e^{i\omega t},$$
$$\phi(x,t)=\sum_{m=1,3,5}^{\infty} \phi_m \cos\alpha x e^{i\omega t},\quad q(x)=\sum_{m=1,3,5}^{\infty} q_m \sin\alpha x\quad , \tag{17}$$

where $i=\sqrt{-1}$, $\alpha = m\pi / L$ and u_m, w_m, ϕ_m are the unknown coefficients; ω is the natural frequency; and q_m is the coefficient of Fourier expansion of the transverse load.

$$q_m = q_0 \qquad \text{for sinusoidal load}$$
$$q_m = \frac{4q_0}{m\pi} \qquad \text{for uniform load}\quad , \tag{18}$$

where q_0 is the maximum intensity of load. By substituting Eqs. (17) and (18) into Eq. (16), the analytical solution can be obtained from the following equations:

$$[K]\times\{\Delta\}=\{Q\}\ , \tag{19}$$

$$\{[K]-N_0[N]\}\times\{\Delta\}=\{0\} \tag{20}$$

$$\{[K]-\omega^2[M]\}\times\{\Delta\}=\{0\} \tag{21}$$

where [K] represents stiffness matrix, [M] represents mass matrix, [N] represents geometric matrix, {Δ} represents vector of unknowns and {Q} represents force vector. The elements of these matrices are as follows:

$$[K]=\begin{bmatrix} A\alpha^2 & -B\alpha^3 & C\alpha^2 \\ -B\alpha^3 & D\alpha^4 & -E\alpha^3 \\ C\alpha^2 & -E\alpha^3 & F\alpha^2+H \end{bmatrix},\quad \{\Delta\}=\begin{Bmatrix} u_m \\ w_m \\ \phi_m \end{Bmatrix},\quad \{Q\}=\begin{Bmatrix} 0 \\ q_m \\ 0 \end{Bmatrix},$$
$$[N]=\begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } [M]=\begin{bmatrix} I_A & -I_B\alpha & I_C \\ -I_B\alpha & (I_D\alpha^2+I_A) & -I_E\alpha \\ I_C & -I_E\alpha & I_F \end{bmatrix}\ , \tag{22}$$

4. Numerical results and discussion

4.1 Numerical results

Numerical results for displacements, stresses, critical buckling load and natural frequencies of functionally graded beams resting on two parameter elastic foundation are presented in this section to verify the accuracy of the present theory. The beam is made of following material properties:

Ceramic: Alumina (Al_2O_3) ($E_c = 380$ GPa, $\rho_c = 3960$ kg/m^3, $\mu = 0.3$)

Metal: Aluminum (Al) ($E_m = 70$ GPa, $\rho_m = 2702$ kg/m^3, $\mu = 0.3$)

For simplicity, the following non-dimensional parameters are used:

Axial displacement ($x = 0, z = -h/2$): $\bar{u} = \dfrac{100u E_m h^3}{q_0 L}$

Transverse displacement ($x = L/2, z = 0$): $\bar{w} = 100\, wE_m h^3 / q_0 L$

Axial stress ($x = L/2, z = h/2$): $\bar{\sigma}_x = \sigma_x h / q_0 L$

Transverse shear stress ($x = 0, z = 0$): $\bar{\tau}_{xz} = \tau_{xz} h / q_0 L$

Critical buckling load: $N_{cr} = 12 N_0 a^2 / E_m h^3$

Fundamental frequency: $\bar{\omega} = \omega L^2 / h\sqrt{\rho_m / E_m}$

4.2 Discussion

In this example, the bending response of FG beam under sinusoidal load is investigated. Displacements and stresses obtained by using the present theory (IHBT), parabolic beam theory (PBT) of Reddy (1984) and first order beam theory (FBT) of Timoshenko (1921) are presented in Table 1. These results are obtained for $L/h = 5$ and various values of power law index (***p***) and foundation parameters (ζ_w, ζ_p). It is seen that the displacements and stresses obtained from the present theory are in excellent agreement with those obtained from PBT. The FBT underestimates the displacements and stresses. Furthermore, it is observed from the Table 1 that the displacements are increased with the increase in power-law index whereas stresses are identical when beam is made of fully ceramic ($p = 0$) or fully metal ($p = \infty$). This is due to the fact that an increase of the power-law index makes FG beams more flexible i.e. reduces the stiffness. It is also observed from the Table 1 that the displacement and stresses of FG beam are reduced when it is resting on two parameter elastic foundation i.e. Winkler layer and shearing layer. Figs. 3-5 show effect of power-law index and foundation parameter on axial displacement of FG beam subjected to sinusoidal load. Figs. 6-8 show non-linear variation of bending stress for ***p*** = 2, 5 and 10and linear variation for ***p*** = 0 and ∞. Through the thickness variations of transverse shear stresses are shown in Figs. 9-11 for various values of power law index and foundation parameters.

The critical buckling loads of FG beams resting on two parameter elastic foundation obtained from the present theory (IHBT), PBT, FBT and Vo *et al*. (2014b) are presented in Table 2. The critical buckling load is obtained for various values of the power-law index (***p***), length to thickness ratio (*L/h*) and foundation parameters. As expected, it is observed that a beam with larger slenderness ratio has a smaller critical buckling load. It can be seen that the increase of ***p*** leads to the reduction in critical buckling load, indicating that the increase in ***p*** decreases the stiffness of beam. It is also observed that the critical buckling load is minimum when beam is resting on two parameter elastic foundation.

The non-dimensional natural frequencies of a simply supported FG beam obtained from the present theory (IHBT) are given in Table 3 for different values of power-law index. The present results are compared with those presented by Reddy (1984), Simsek (2010), Thai and Vo (2012), Vo *et al*. (2014a), Timoshenko (1921) and Bernoulli-Euler. The examination of Table 3 reveals that the fundamental frequencies obtained using the present theory are in excellent agreement with the previously published results. It is observed that an increase in value of the ***p*** leads to a reduction of fundamental frequencies. This is due to an increase in value of ***p*** results in a decrease in the value of elasticity modulus. The non-dimensional natural frequencies of an FG beam resting on two parameter elastic foundation obtained from the present theory (IHBT) are given in Table 4 for different values of power-law index. These results are first time presented in this paper. It is observed that the natural frequencies are increased when beam is resting on two parameter elastic foundation.

Table 1 Non-dimensional displacements and stresses of functionally graded beam resting on two parameter elastic foundation and subjected to sinusoidal load

				L/h = 5				L/h = 20			
p	ξ_w	ξ_p	Theory	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}$	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}$
0	0	0	Present	0.7253	2.5019	3.0922	0.4800	0.1784	2.2839	12.171	0.4806
			Reddy (1984)	0.7251	2.5020	3.0916	0.4769	0.1784	2.2838	12.171	0.4774
			Timoshenko (1921)	0.7129	2.0523	3.0396	0.2653	0.1782	2.2839	12.158	0.2653
	0.1	0	Present	0.6826	2.3547	2.9102	0.4517	0.0932	1.1935	6.3608	0.2511
			Reddy (1984)	0.6824	2.3547	2.9096	0.4488	0.0932	1.1935	6.3606	0.2495
			Timoshenko (1921)	0.6716	2.3205	2.8607	0.2499	0.0932	1.1929	6.3539	0.1387
	0.1	0.1	Present	0.4317	1.4894	1.8407	0.2857	0.0163	0.2090	1.1136	0.0440
			Reddy (1984)	0.4316	1.4894	1.8403	0.2839	0.0163	0.2090	1.1136	0.0437
			Timoshenko (1921)	0.4271	1.4756	1.8093	0.1589	0.0163	0.2089	1.1124	0.0243
1	0	0	Present	1.7796	4.9441	4.7867	0.5248	0.4400	4.5774	18.814	0.5245
			Reddy (1984)	1.7793	4.9458	4.7857	0.5243	0.4400	4.5773	18.813	0.5249
			Timoshenko (1921)	1.7588	4.8807	4.6979	0.5376	0.4397	4.5734	18.792	0.5376
	0.1	0	Present	1.5838	4.4015	4.2600	0.4657	0.1554	1.6169	6.6458	0.1851
			Reddy (1984)	1.5835	4.4015	4.2591	0.4666	0.1554	1.6169	6.6456	0.1854
			Timoshenko (1921)	1.5675	4.3499	4.1871	0.4791	0.1554	1.6164	6.6418	0.1900
	0.1	0.1	Present	0.7592	2.1100	2.0422	0.2232	0.0211	0.2190	0.9001	0.0251
			Reddy (1984)	0.7591	2.1100	2.0417	0.2237	0.0211	0.2190	0.9001	0.0251
			Timoshenko (1921)	0.7560	2.0981	2.0195	0.2311	0.0211	0.2190	0.8998	0.0257
5	0	0	Present	2.8649	7.7739	6.6079	0.5274	0.7069	6.9541	25.795	0.5313
			Reddy (1984)	2.8644	7.7723	6.6057	0.5314	0.7069	6.9540	25.794	0.5323
			Timoshenko (1921)	2.8250	7.5056	6.4382	0.9942	0.7062	6.9373	25.752	0.9942
	0.1	0	Present	2.3987	6.5089	5.5327	0.4416	0.1869	1.8389	6.8212	0.1397
			Reddy (1984)	2.3984	6.5078	5.5310	0.4450	0.1869	1.8389	6.8211	0.1408
			Timoshenko (1921)	2.3786	6.3198	5.4210	0.8371	0.1871	1.8377	6.8221	0.2634

Table 1 (Continued)

	0.1	0.1	Present	0.9205	2.4976	2.1231	0.1694	0.0226	0.2226	0.8258	0.0170
			Reddy (1984)	0.9204	2.4975	2.1226	0.1708	0.0226	0.2226	0.8258	0.0170
			Timoshenko (1921)	0.9294	2.4693	2.1181	0.3271	0.0227	0.2226	0.8264	0.0319
10	0	0	Present	2.9995	8.6539	7.9102	0.4237	0.7380	7.6422	30.923	0.4263
			Reddy (1984)	2.9989	8.6530	7.9080	0.4226	0.7379	7.6421	30.999	0.4233
			Timoshenko (1921)	2.9488	8.3259	7.7189	1.2320	0.7372	7.6215	30.875	1.2320
	0.1	0	Present	2.4659	7.1147	6.5033	0.3484	0.1819	1.8838	7.6225	0.1051
			Reddy (1984)	2.4655	7.1141	6.5016	0.3474	0.1819	1.8838	7.5606	0.1043
			Timoshenko (1921)	2.4408	6.8914	6.3891	1.0197	0.1821	1.8825	7.6262	0.3043
	0.1	0.1	Present	0.8950	2.5820	2.3601	0.1264	0.0216	0.2233	0.9035	0.0125
			Reddy (1984)	0.8948	2.5819	2.3596	0.1261	0.0215	0.2233	0.8934	0.0124
			Timoshenko (1921)	0.9039	2.5520	2.3660	0.3776	0.0216	0.2233	0.9045	0.0361
∞	0	0	Present	3.9371	13.582	3.0922	0.4800	0.9677	12.329	12.171	0.4806
			Reddy (1984)	3.9363	13.582	3.0916	0.4769	0.9686	12.398	12.171	0.4774
			Timoshenko (1921)	3.8702	12.552	3.0396	0.3183	0.9676	12.398	12.158	0.3183
	0.1	0	Present	2.9391	10.139	2.3084	0.3583	0.1631	2.0785	2.0425	0.0806
			Reddy (1984)	2.9385	10.139	2.3079	0.3560	0.1625	2.0805	2.0424	0.0801
			Timoshenko (1921)	2.8891	10.140	2.2691	0.2376	0.1624	2.0805	2.0403	0.0534
	0.1	0.1	Present	0.8393	2.8955	0.6592	0.1023	0.0177	0.2258	0.2217	0.0088
			Reddy (1984)	0.8392	2.8955	0.6591	0.1017	0.0176	0.2258	0.2217	0.0087
			Timoshenko (1921)	0.8250	2.8955	0.6479	0.0679	0.0176	0.2258	0.2214	0.0058

Table 2 Non-dimensional critical buckling load of functionally graded beam resting on two parameter elastic foundation

				p					
L/h	ξ_w	ξ_p	Theory	0	1	2	5	10	∞
5	0	0	Present	48.596	24.584	19.070	15.640	14.049	8.9520
			Reddy (1984)	48.596	24.584	19.071	15.643	14.050	9.3968
			Vo *et al*. (2014b)	48.840	24.691	19.160	15.740	14.146	---
			Timoshenko (1921)	49.356	24.911	19.422	16.199	14.603	9.0915
	0.1	0	Present	51.636	27.623	22.110	18.680	17.089	11.991
			Reddy (1984)	51.635	27.623	22.110	18.683	17.090	12.436
			Timoshenko (1921)	52.395	27.951	22.461	19.238	17.643	12.131
	0.1	0.1	Present	81.636	57.623	52.110	48.680	47.089	41.991
			Reddy (1984)	81.635	57.623	52.110	48.683	47.090	42.436
			Timoshenko (1921)	82.395	57.951	52.461	49.238	47.643	42.131

Table 2 (Continued)

10	0	0	Present	52.237	26.140	20.366	17.080	15.498	9.6227
			Reddy (1984)	52.238	26.140	20.366	17.081	15.499	9.7375
			Vo *et al*. (2014b)	52.308	26.172	20.393	17.111	15.529	---
			Timoshenko (1921)	52.456	26.233	20.465	17.243	15.663	9.6629
	0.1	0	Present	64.396	38.299	32.524	29.239	27.657	21.781
			Reddy (1984)	64.396	38.299	32.524	29.240	27.657	21.895
			Timoshenko (1921)	64.614	38.391	32.623	29.402	27.821	21.821
	0.1	0.1	Present	184.39	158.29	152.52	149.24	147.65	141.78
			Reddy (1984)	184.39	158.29	152.52	149.24	147.65	141.89
			Timoshenko (1921)	184.61	158.39	152.62	149.40	147.82	141.82
20	0	0	Present	53.236	26.562	20.718	17.484	15.909	9.8066
			Reddy (1984)	53.236	26.562	20.718	17.484	15.909	9.8356
			Timoshenko (1921)	53.292	26.585	20.743	17.526	15.952	9.8170
	0.1	0	Present	101.87	75.196	69.352	66.118	64.544	58.440
			Reddy (1984)	101.87	75.196	69.352	66.118	64.544	58.469
			Timoshenko (1921)	101.92	75.219	69.377	66.160	64.587	58.451
	0.1	0.1	Present	581.87	555.19	549.35	546.11	544.54	538.44
			Reddy (1984)	581.87	555.19	549.35	546.11	544.54	538.46
			Timoshenko (1921)	581.92	555.21	549.37	546.16	544.58	538.45
50	0	0	Present	53.523	26.682	20.819	17.601	16.029	9.8594
			Reddy (1984)	53.519	26.687	20.820	17.602	16.029	9.8647
			Timoshenko (1921)	53.532	26.686	20.825	17.608	16.037	9.8600
	0.1	0	Present	357.48	330.64	324.78	321.56	319.99	313.82
			Reddy (1984)	357.48	330.64	324.78	321.56	319.99	313.82
			Timoshenko (1921)	357.49	330.65	324.79	321.57	320.00	313.82
	0.1	0.1	Present	3357.4	3330.6	3324.7	3321.5	3319.9	3313.8
			Reddy (1984)	3357.4	3330.6	3324.7	3321.5	3319.9	3313.8
			Timoshenko (1921)	3357.5	3330.6	3324.7	3321.5	3320.0	3313.8
100	0	0	Present	53.565	26.700	20.835	17.619	16.046	9.8670
			Reddy (1984)	53.565	26.704	20.837	17.620	16.048	9.8688
			Timoshenko (1921)	53.581	26.697	20.845	17.618	16.053	9.8652
	0.1	0	Present	1269.4	1242.5	1236.6	1233.4	1231.9	1225.7
			Reddy (1984)	1269.4	1242.5	1236.6	1233.4	1231.9	1225.7
			Timoshenko (1921)	1269.4	1242.5	1236.6	1233.4	1231.9	1225.7
	0.1	0.1	Present	13269.4	13242.5	13236.6	13233.4	13231.9	13225.7
			Reddy (1984)	13269.4	13242.5	13236.6	13233.4	13231.9	13225.7
			Timoshenko (1921)	13269.4	13242.5	13236.6	13233.4	13231.9	13225.7

Table 3 Non-dimensional natural frequencies of simply supported functionally graded beam

		p					
L/h	Theory	0	1	2	5	10	∞
5	Present	5.1453	3.9826	3.6184	3.3917	3.2727	2.6734
	Reddy (1984)	5.1527	3.9904	3.6264	3.4012	3.2816	2.6773
	Simsek (2010)	5.1527	3.9904	3.6261	3.4012	3.2816	2.6773
	Thai and Vo (2012)	5.1527	3.9904	3.6264	3.4012	3.2816	2.6773
	Vo *et al*. (2014a)	5.1527	3.9716	3.5979	3.3742	3.2653	2.6773
	Timoshenko (1921)	5.1524	3.9902	3.6343	3.4311	3.3134	2.6771
	Bernoulli-Euler	5.3953	4.1484	3.7793	3.5949	3.4921	2.8033
20	Present	5.4603	4.2050	3.8361	3.6485	3.5389	2.8371
	Reddy (1984)	5.4603	4.2050	3.8361	3.6485	3.5389	2.8371
	Simsek (2010)	5.4603	4.2050	3.8361	3.6485	3.5389	2.8371
	Thai and Vo (2012)	5.4603	4.2050	3.8361	3.6484	3.5389	2.8371
	Vo *et al*. (2014a)	5.4603	4.2038	3.8342	3.6466	3.5378	2.8371
	Timoshenko (1921)	5.4603	4.2050	3.8367	3.6508	3.5415	2.8371
	Bernoulli-Euler	5.4777	4.2163	3.8472	3.6628	3.5547	2.8461

Table 4 Non-dimensional flexural natural frequencies of functionally graded beams resting on elastic foundation

				p					
L/h	Mode	ξ_w	ξ_p	0	1	2	5	10	∞
5	1	0	0	5.1453	3.9826	3.6184	3.3917	3.2727	2.6734
		0.1	0	5.3038	4.2216	3.8961	3.7066	3.6094	3.0942
		0.1	0.1	6.6689	6.0973	5.9810	5.9830	5.9909	5.7903
	2	0	0	17.589	13.754	12.388	11.260	10.748	9.1392
		0.1	0	17.633	13.820	12.465	11.351	10.848	9.2623
		0.1	0.1	19.287	16.200	15.200	14.493	14.224	13.240
	3	0	0	32.324	25.538	22.812	20.117	19.003	16.794
		0.1	0	32.346	25.570	22.849	20.163	19.053	16.855
		0.1	0.1	34.223	28.261	25.980	23.881	23.1.7	21.626
20	1	0	0	5.4603	4.2050	3.8361	3.6484	3.5389	2.8371
		0.1	0	7.5533	7.0751	7.0184	7.0948	7.1279	6.9259
		0.1	0.1	18.052	19.224	19.752	20.390	20.703	21.022
	2	0	0	21.571	16.631	15.158	14.370	13.922	11.208
		0.1	0	22.189	17.571	16.250	15.596	15.226	12.857
		0.1	0.1	39.513	39.730	40.223	41.157	41.624	41.615
	3	0	0	47.569	36.740	33.440	31.543	30.505	24.716
		0.1	0	47.851	37.171	33.943	32.114	31.116	25.499
		0.1	0.1	68.353	64.871	64.523	65.245	65.633	64.358

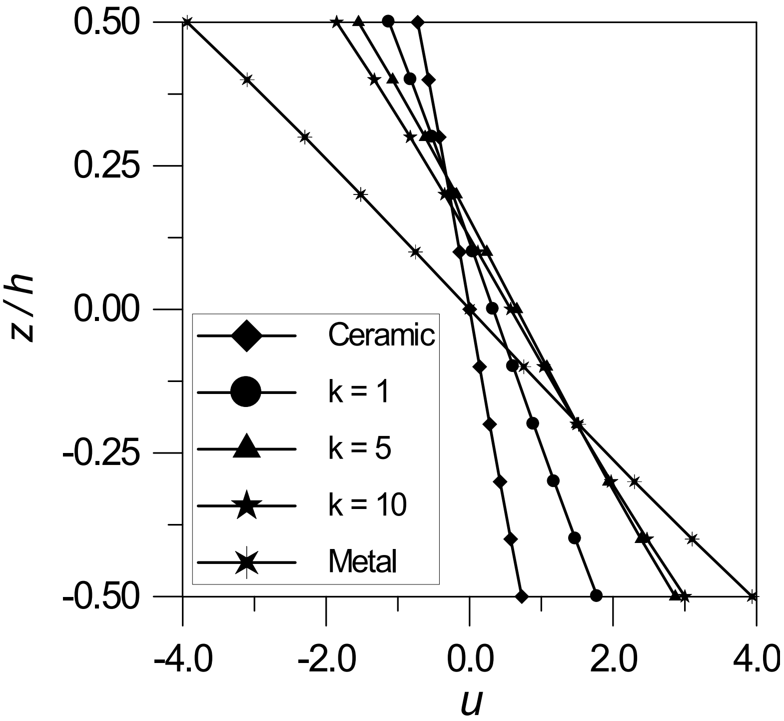

Fig. 3 Non-dimensional axial displacement through the thickness (L/h = 5)

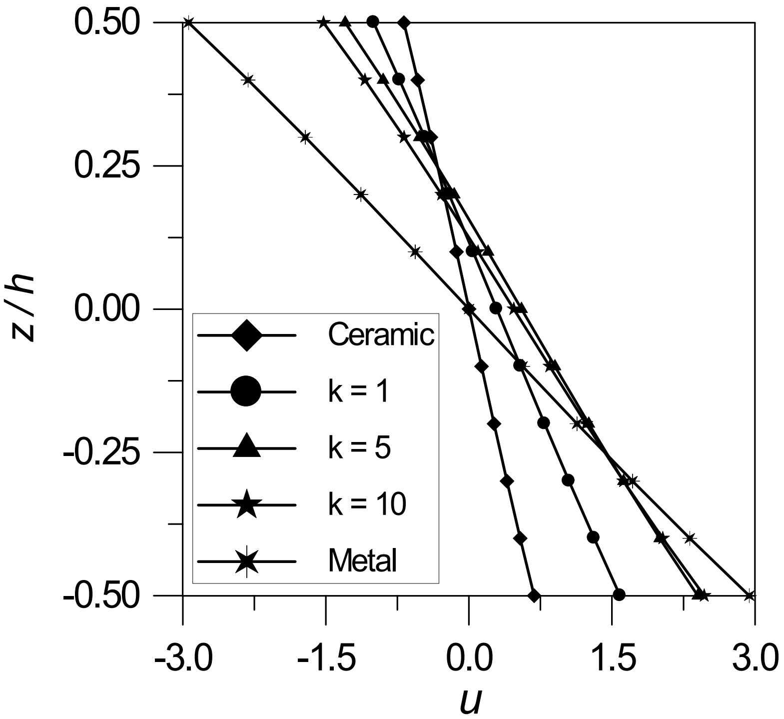

Fig. 4 Non-dimensional axial displacement through the thickness (L/h = 5)

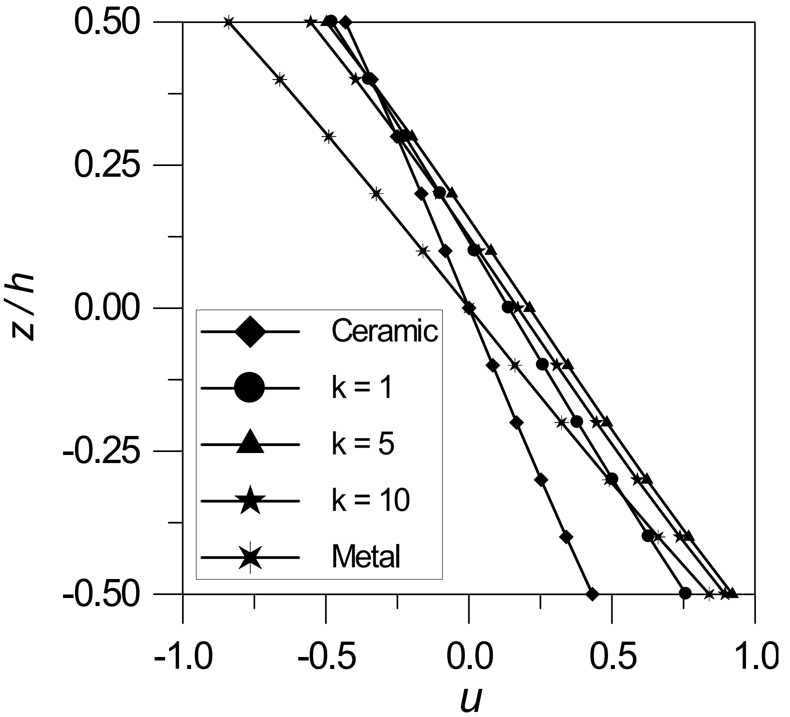

Fig. 5 Non-dimensional axial displacement through the thickness (*L*/*h* = 5)

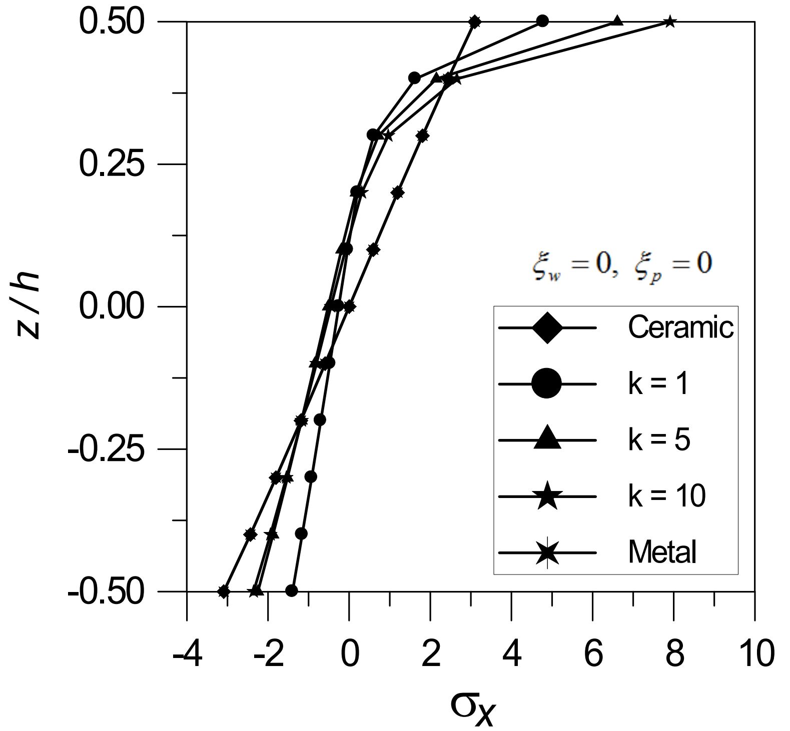

Fig. 6 Non-dimensional axial stress through the thickness (*L*/*h* = 5)

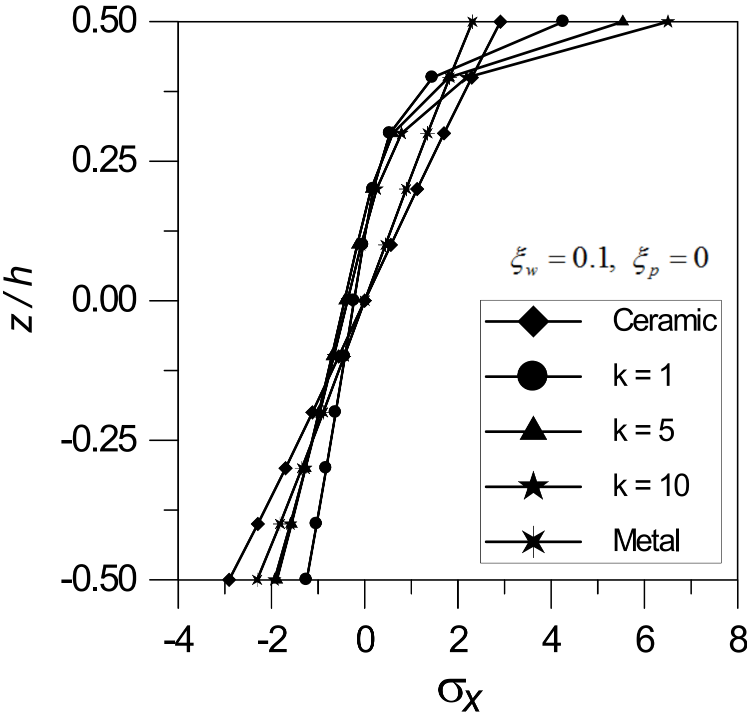

Fig. 7 Non-dimensional axial stress through the thickness (*L/h* = 5)

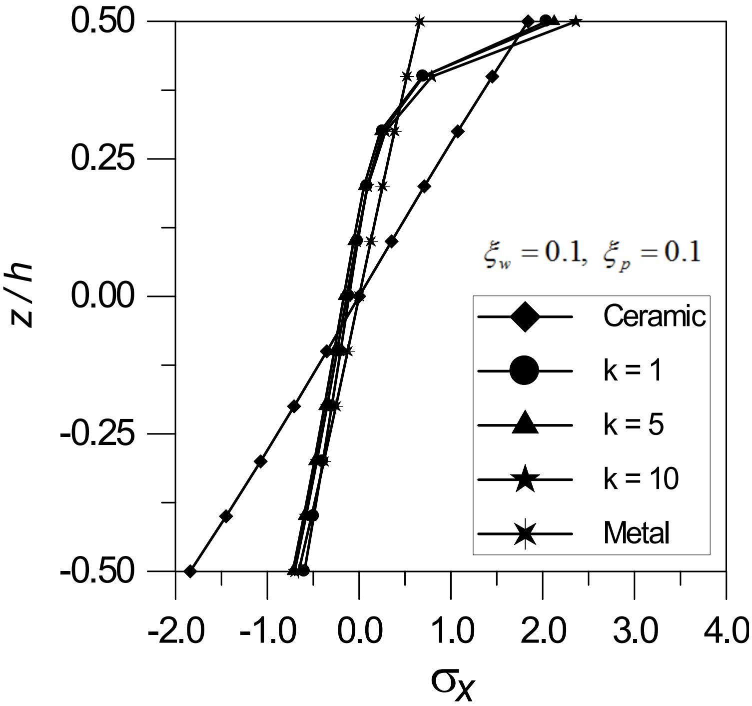

Fig. 8 Non-dimensional axial stress through the thickness (*L/h* = 5)

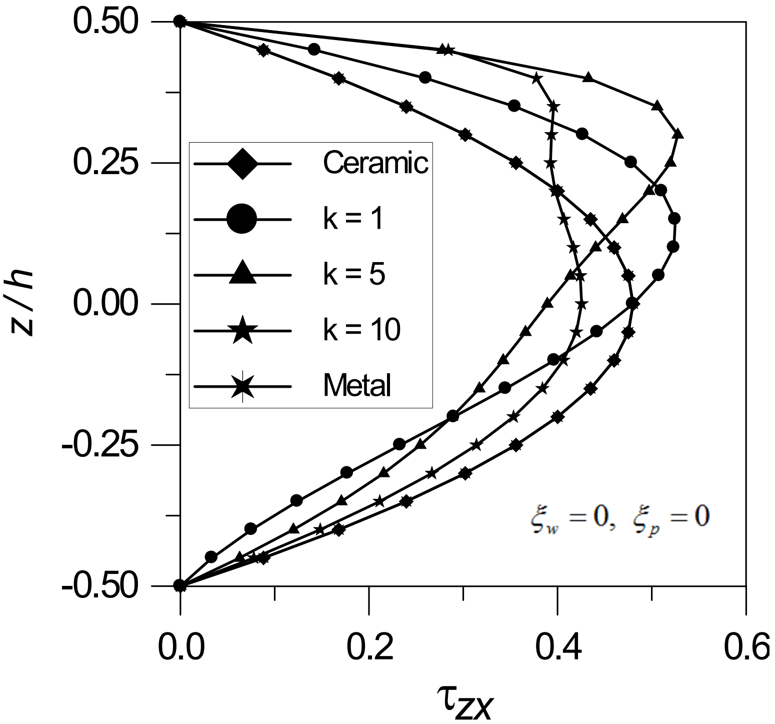

Fig. 9 Non-dimensional transverse shear stress through the thickness (*L/h* = 5)

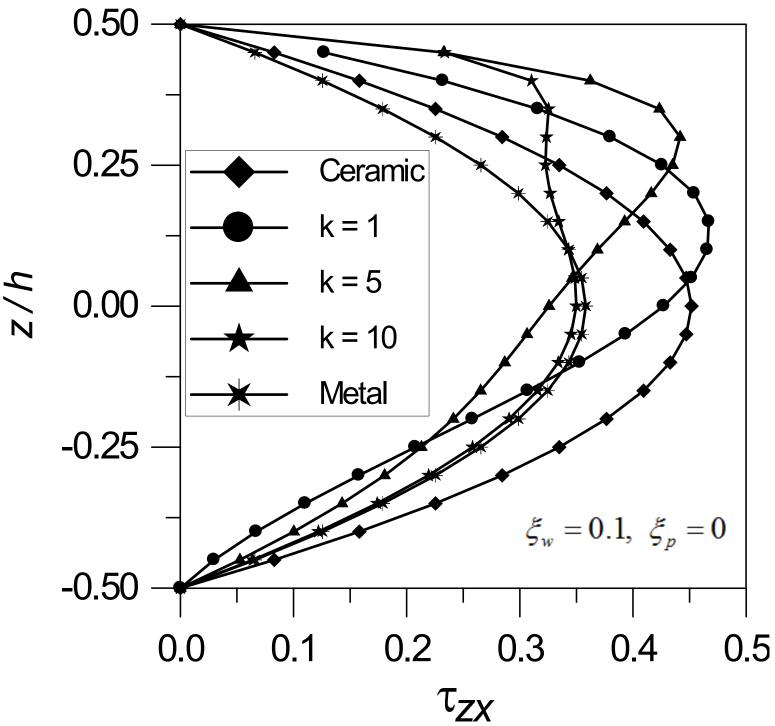

Fig. 10 Non-dimensional transverse shear stress through the thickness (*L/h* = 5)

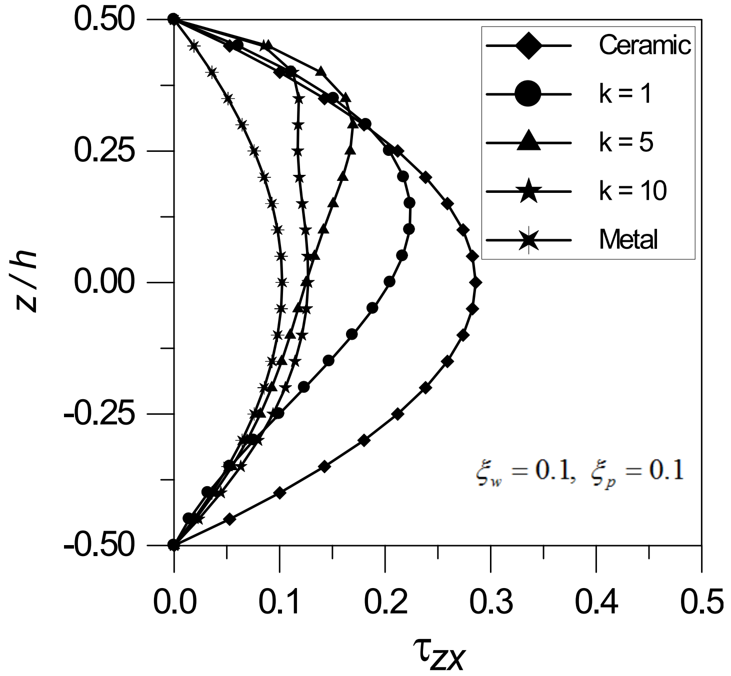

Fig. 11 Non-dimensional transverse shear stress through the thickness (L/h = 5)

5. Conclusions

Analytical solutions for bending, buckling and free vibration analysis of functionally graded beams resting on two parameter elastic foundation are presented in this study. An inverse hyperbolic shear deformation theory taking into account effect of transverse shear deformation is presented. The theory gives hyperbolic cosine variation of transverse shear stress across the thickness of the beam. Effects of the power-law index, length-to-thickness ratio and foundation parameter on the displacements, stresses, critical buckling loads and natural frequencies of FG beams are investigated. From the numerical results, it is concluded that the increase in the power-law index reduces the stiffness of functionally graded beam and consequently leads to an increase in displacements and a reduction of natural frequencies and critical buckling loads. The proposed theory is accurate and efficient in predicting bending, buckling and free vibration responses of functionally graded beams. Natural frequencies of FG beams resting on elastic foundation are presented for the first time in the present study and can be served as a benchmark solution for the future researchers.

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