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Condensation of independent variables in free vibration analysis of curved beams

Yusuke Mochida* and Sinniah Ilanko

The School of Engineering, The University of Waikato, Private Bag 3105, Hamilton 3240, New Zealand

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Abstract. In this paper, the condensation method which is based on the Rayleigh-Ritz method, is described for the free vibration analysis of axially loaded slightly curved beams subject to partial axial restraints. If the longitudinal inertia is neglected, some of the Rayleigh-Ritz minimization equations are independent of the frequency. These equations can be used to formulate a relationship between the weighting coefficients associated with the lateral and longitudinal displacements, which leads to "connection coefficient matrix". Once this matrix is formed, it is then substituted into the remaining Rayleigh-Ritz equations to obtain an eigenvalue equation with a reduced matrix size. This method has been applied to simply supported and partially clamped beams with three different shapes of imperfection. The results indicate that for small imperfections resembling the fundamental vibration mode, the sum of the square of the fundamental natural and a non-dimensional axial load ratio normalized with respect to the fundamental critical load is approximately proportional to the square of the central displacement.

Keywords: condensation method; connection coefficient matrix; slightly curved beam; natural frequencies; axial load

1. Introduction

Some eigenvalue problems can be formulated based on variational methods in terms of two or more independent field variables but simplifications make it possible to reduce the matrix size by formulating the relationship between two or more of the field variables that are independent of the eigenvalue and using this relationship in the variational equation. This method has already been used in post-buckling and vibration behaviour of in-plane loaded rectangular plates with geometric out-of-plane initial imperfections by one of the authors two decades ago (Ilanko and Dickinson 1987 and Ilanko 2002) but it does not appear to have been widely used since then, possibly because the focus of the publications that used this approach were more on the vibratory behaviour. This paper shows this method through application to a simpler problem, namely the linear eigenvalue equations for the lateral vibration of slightly curved beams. It is shown how the Rayleigh-Ritz minimisation, together with the condensation method can be used to find the natural

*Corresponding author, Ph.D., E-mail: yusuke@waikato.ac.nz
[a]Professor, E-mail: ilanko@waikato.ac.nz

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frequencies of slightly curved beams with results for simply supported and clamped beams having different shapes of initial imperfection subject to axial force. The beam is assumed to be slender and the longitudinal inertia is neglected and the standard assumptions for Euler-Bernoulli beams are also made. The Rayleigh-Ritz minimisation equations are obtained in terms of the coordinates associated with the two field variables namely, longitudinal and lateral displacements. These displacements forms are expressed as a product of a series of admissible functions and undetermined weighting coefficients the Rayleigh-Ritz minimisation is first carried out with respect to the weighting coefficients. Then the frequency independent equations are solved analytically to obtain a relationship between the coefficients associated with the longitudinal and lateral displacement series. This relationship is then substituted into the remaining frequency dependent Rayleigh-Ritz equations to obtain am eigenvalue equation in which the eigenvectors are the lateral displacement coefficients.

The effect of geometrical imperfections on the flexural natural frequencies of beams has been reported in (Kim and Dickinson 1986, Ilanko 1990 and Ilanko and Dickinson 1987). Kim and Dickinson (1986) presented experimental and theoretical results for slightly curve beams under static axial force. The influence of partial axial end restraints on the natural frequencies of simply supported beams using a Newtonian approach is discussed in (Ilanko 1990) In this paper, a general procedure based on the Rayleigh-Ritz method is described, for the free vibration analysis of axially loaded slightly curved beams subject to partial axial restraints. Initially, both lateral and longitudinal dynamic displacements are expressed as the summation of the product of permissible functions and undetermined weighting coefficients. The Rayleigh-Ritz minimization equations are then obtained by minimizing the Rayleigh quotient with respect to all undetermined coefficients. Neglecting the longitudinal inertia of the beam yields some frequency independent equations (those obtained by minimizing with respect to the weighting coefficients associated with longitudinal motion). This set of equations is then reformulated to obtain the frequency independent relationship between the longitudinal and lateral displacement coefficients. This approach was used to analyse the post buckling and vibration behaviour of plates (Ilanko and Dickinson 1987), in which a matrix called “connection coefficient matrix” gives the relationship between the out-of-plane and in-plane displacement coefficients. A similar connection coefficient matrix is used in the present work to express the relationship between the longitudinal and lateral displacement coefficients. Once this matrix is formed, it is then substituted into the remaining Rayleigh-Ritz equations to obtain an eigenvalue equation with a reduced matrix size. This method has been applied to simply supported and partially clamped beams with three different shapes of imperfection. The results indicate that for small imperfections resembling the fundamental vibration mode, the sum of the square of the fundamental natural frequency and a non-dimensional axial load ratio normalized with respect to the fundamental critical load is approximately proportional to the square of the central displacement.

2. Theory

Consider the small amplitude flexural vibration of an axially loaded, partially restrained Euler-Bernoulli beam with initial curvature $v_0(x)$ as shown in Fig. 1. The vibration is assumed to be in the plane containing the curvature. The beam has flexural rigidity EI, cross sectional area A, length L and density ρ. Let $v(x)$ be the deflected shape of the beam under static compressive axial force P. For vibration in a principal mode, the dynamic lateral displacement (measured from the static

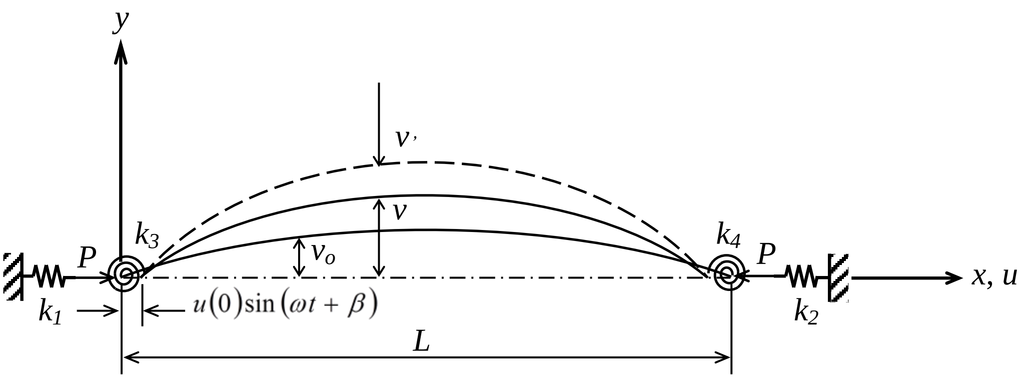


Fig. 1 Laterally vibrating beam

equilibrium position) may be written as $v'(x)\ \sin(\omega t+\beta)$, where ω is a natural frequency. The corresponding longitudinal dynamic displacement is $u(x)\ \sin(\omega t+\beta)$. It is assumed that the vibration is primarily lateral.

Let the dynamic displacement measured from the static equilibrium state be

$$v'(x) = \sum_{i=1,2..} d_i \phi_i(x) \tag{1}$$

where $\phi_i(x)$ are permissible displacement forms that satisfy the geometrical constraints on the beam. The maximum total potential energy of the beam is given by

$$V = U_b + U_a + V_a + V_s + V_b + V_e \tag{2}$$

in which the energy terms U_b etc are defined as follows:

The maximum strain energy due to dynamic bending is

$$U_b = {}_0^L (EI/2)(\partial^2 v'/\partial x^2)^2 \mathrm{d}x \tag{2a}$$

The maximum strain energy associated with the non-flexural axial straining is (Ilanko and Dickinson 1987)

$$U_a = \int_0^L (EA/2)[\partial u/\partial x + (\partial v/\partial x)(\partial v'/\partial x)]^2 \mathrm{d}x \tag{2b}$$

The maximum potential energy due to the static axial force is

$$V_a = -\int_0^L (P/2)(\partial v?/\partial x)^2 \mathrm{d}x \tag{2c}$$

The maximum potential energy due to the longitudinal springs at the ends is

$$V_S = (1/2)k_1 u^2(0) + (1/2)k_2 u^2(L) \tag{2d}$$

For beams partially or fully restrained against rotation the following additional term is need

$$V_b = 1/2\left(k_3 \left(v'(0)\right)^2 + k_4 \left(v'(L)\right)^2\right) \tag{2e}$$

in which, k_3, k_4 are the rotational stiffnesses of the partial restraints and v' is the slope of the maximum lateral displacement. By setting these stiffness coefficients to zero or very high values, simply supported and laterally clamped boundary conditions may be obtained.

V_e is the total potential energy of the statically loaded beam at equilibrium.

Let the dynamic longitudinal displacement at the time of maximum excursion be

$$u(x) = \sum_{m=1,2..} \alpha_m \chi_m (x) \tag{3}$$

Neglecting the longitudinal inertia, the maximum kinetic energy is given by

$$T = \omega^2 \psi \tag{4}$$

where ψ is a kinetic energy function which is defined by

$$\psi = (1/2)\int_0^L \rho A v'^2 \, \mathrm{d}x \tag{4a}$$

Using the Rayleigh-Ritz procedure, the natural frequencies are obtained by solving the following equations

$$\{\partial V / \partial \alpha_m\} = \{0\} \tag{5a}$$

$$\{(\partial V / \partial d_i) - \omega^2 (\partial \psi / \partial d_i)\} = \{0\} \tag{5b}$$

If there are n_α number of terms in u and n_d number of terms in the series for v', the total number of equations is $n_\alpha + n_d$. However, the problem can be condensed by solving it in two steps reducing the number of eigenvalue equations to n_d.

Eq. (5a) yields a relationship between α and d. This may be written in matrix form as

$$\{\alpha\} = [C]\,\{d\} \tag{6}$$

where $[C]$ may be called a connection coefficient matrix since it connects the longitudinal and lateral displacement coefficients. Similar matrices have been used in the post-buckling and vibration analysis of imperfect plates (Ilanko and Dickinson 1987). Substituting Eq. (6) into Eq. (5b) results in an equation of the form

$$[K]\{d\} - \omega^2[M]\{d\} = \{0\} \tag{7}$$

where $[K]$ is a stiffness matrix and $[M]$ is a mass matrix.

The number of simultaneous eigenvalue equations has been reduced from the total number of weighting coefficients to the number of coefficients associated with lateral displacement.

3. Application

For simply supported beams, the initial imperfection and all lateral deflections may be written in the form of a sine series. Initially all series are taken as having infinite number of terms.

$$v_0 = \sum_{i=1,2..} a_i sin(i\pi x / L) \tag{8}$$

The lateral static deflection is given by (Home and Merchant 1965)

$$v = \sum_{i=1,2..} b_i sin(i\pi x / L) \tag{9}$$

where

$$b_i = a_i/(1-P/P_{ci}) \tag{9a}$$

in which P_{ci} is the i^{th} critical load and is given by

$$P_{ci} = EI(i\pi/L)^2 \tag{9b}$$

The lateral dynamic deflection at the time of maximum excursion may be written as

$$v' = \sum_{i=1,2..} d_i sin(i\pi x / L) \tag{10}$$

The sine functions satisfy the geometrical constraints that the lateral displacement v'=0 at x=0 and at x=L, and are therefore permissible in a Rayleigh-Ritz formulation. Similarly permissible functions are required for the longitudinal displacement $u(x)$. For partially longitudinally restrained beams, since no geometrical constraints exist, any continuous function would be permissible. The Newtonian approach used in reference (Ilanko and Dickinson 1986) yielded a combination of a linear function of x and a sine series. Therefore the same type of functions are used in the present analysis. Let

$$u(x) = (\alpha_1 + \alpha_2 x + \textstyle\sum_{m=1,2}\alpha_{m+2} \sin (m\pi x/L)) \tag{11}$$

The maximum strain energy due to dynamic bending is

$$U_b = \frac{\pi^4 EI}{4L^3} \textstyle\sum_{i=1,2..} i^4 d_i^2 \tag{12a}$$

The maximum strain energy due to the non flexural axial straining is

$$U_a = \frac{EA}{2}\int_0^L (\alpha_2 + \frac{\pi}{L}\textstyle\sum_m m\,\alpha_{m+2}\cos(\frac{m\pi x}{L})) + \frac{\pi^2}{L^2}\sum_j\sum_i j i d_i b_j \cos(\frac{j\pi x}{L})\cos(\frac{i\pi x}{L}))^2 \mathrm{d}x \tag{12b}$$

The maximum potential energy associated with the axial force is

$$V_a = -\frac{\pi^2 P}{4L}\textstyle\sum_{i=1,2..} i^2 d_i^2 \tag{12c}$$

The maximum potential energy due to the longitudinal supports is

$$V_S = (1/2)(k_1\alpha_1^2 + k_2(\alpha_1 + \alpha_2 L)^2) \tag{12d}$$

The kinetic energy function ψ is

$$\psi = \frac{\rho AL}{4}\textstyle\sum_i d_i^2 \tag{13}$$

The derivatives of the energy terms with respect to the weighting coefficients are as follows

$$\partial U_b/\partial\alpha_m = 0;\ \partial V_a/\partial\alpha_m = 0;\ \partial V_b/\partial\alpha_m = 0;\ \partial V/\partial\alpha_m = 0, \text{ for all } m, \tag{14a-d}$$

$$\partial U_a/\partial\alpha_1 = 0;\ \partial V_S/\partial\alpha_1 = (k_1 + k_2)\,\partial_1 + k_2 L\alpha_2, \text{ for } m=1 \tag{14e, f}$$

$$\partial U_a/\partial\alpha_2 = EA\alpha_2 L + (EA\pi^2/4L)\textstyle\sum_i i^2 b_i d_i;\ \partial V_S/\partial\alpha_2 = k_2 L(\alpha_1+\alpha_2 L) \text{ for } m = 2, \tag{14g, h}$$

And finally for m>2

$$\partial V_S/\partial \alpha_m = 0; \tag{14i}$$

$$\partial U_a/\partial \alpha_m = EA\alpha_m((m\text{-}2)^2\pi^2/2L) + ((m\text{-}2)EA\pi^3/L^3)\textstyle\sum_j\sum_i ij \int d_i b_j \cos((m\text{-}2)\pi x/L)\cos(i\pi x/L)\cos(j\pi x/L)\,\mathrm{d}x \tag{14j}$$

Substituting Eqs. (14a-j) into Eq. (5a) yields the following relationship between α and d

$$\{\alpha\} = [\mathrm{C}]\ \{d\} \tag{15}$$

where the elements of the connection coefficients matrix [C] are given by

$$C_{1i} = \frac{k_2 EAi^2\pi^2 b_i}{4(k_1k_2L^2 + (k_1+k_2)EAL)}, \tag{15a}$$

$$C_{2i} = -\frac{(k_1+k_2)EAi^2\pi^2 b_i}{4(k_1k_2L^2 + (k_1+k_2)EAL)L}, \tag{15b}$$

and for m>2

$$C_{mi} = -(2\pi i/(m\text{-}2)L^2)\textstyle\sum_j j\, b_j \int \cos((m\text{-}2)\,\pi x/L)\cos(i\pi x/L)\cos(j\pi x/L)\,\mathrm{d}x, \tag{15c}$$

In all derivations up to Eq. (14j), the deflections were taken as infinite sine series. For actual calculations the series may be truncated. If there are n_b number of terms in the series for initial imperfection (and hence for the static deflection v), and n_d terms in the series for v', then the integers j and i in Eq. (15c) would be limited to n_b and n_d respectively. Since the integral in Eq. (15c) would be zero for m-2>n_b+n_d, the maximum number of longitudinal displacement coefficients n may be set to n_b+n_d+2. [C] is a matrix of size $n\times n_d$.

At this stage minimization with respect to d_m may be carried out. This yields

$$\frac{\partial U_b}{\partial d_i} = \frac{i^4\pi^4 EI}{2L^3} d_i \tag{16a}$$

$$\partial V_b/\partial d_i = \sum_m \big(k_3 + k_4\cos(i\pi)\cos(m\pi)\big)d_m \tag{16b}$$

$$\partial V_S/\partial d_i = \partial\ V_e/\partial d_i = 0; \tag{16c, d}$$

$$\partial V_a/\partial d_i = -(i^2\pi^2P/(2L))d_i \tag{16e}$$

$$\frac{\partial U_a}{\partial d_i} = \frac{EA\pi^2 i^2 b_i\alpha_2}{2L} + \frac{iEA\pi^3}{L^3}\textstyle\sum_m m\,\alpha_{m+2}\sum_j j\,b_j\int_0^L \cos(\frac{i\pi x}{L})\cos(\frac{j\pi x}{L})\cos(\frac{m\pi x}{L})dx$$

$$+\ \frac{EA\pi^4}{L^4}\textstyle\sum_j ij b_j\sum_m\sum_n nm b_m d_n\int_0^L \cos(\frac{i\pi x}{L})\cos(\frac{j\pi x}{L})\cos(\frac{m\pi x}{L})\cos(\frac{n\pi x}{L})dx, \tag{16f}$$

Also

$$\partial\psi/\partial d_i = \tfrac{1}{2}\,\rho AL\, d_i \tag{16g}$$

Substituting Eqs. (16a-g) into Eq. (5b) gives

$$\left(\frac{i^4\pi^4 EI}{2L^3} - \frac{i^2\pi^2 P}{2L} - \frac{\rho AL\omega^2}{2}\right)d_i + \sum_m \alpha_m Q_{mi} +$$

$$\frac{EA\pi^4}{L^4}\sum_j\sum_m\sum_n ijmn\, b_m b_j d_n \int_0^L \cos\left(\frac{i\pi x}{L}\right)\cos\left(\frac{j\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0 \qquad (17)$$

where

$$Q_{1i} = 0 \qquad (17a)$$

$$Q_{2i} = ½\, EA\pi^2 m^2 b_i/L \qquad (17b)$$

and for m>2

$$Q_{mi} = (m\text{-}2)i(EA\pi^3/L^3)\textstyle\sum_j j\, b_j\int_0^L \cos((m\text{-}2)\pi x/L)\cos(j\pi x/L)\cos(i\pi x/L)\, dx \qquad (17c)$$

Substituting Eq. (15) into the above equation, and some lengthy algebraic and trigonometrical manipulations result in the following equation

$$(EI(i\pi/L)^4 - (i\pi/L)^2P - \rho A\omega^2)\, d_i + k_e\, (\pi^4/2L^3)\, b_i\, i^2 \textstyle\sum_{j=1,2..} j^2 b_j d_j = 0, \qquad (18)$$

where k_e is an effective end stiffness parameter given by

$$k_e = 1/(1/k_1+1/k_2+L/(EA)) \qquad (18a)$$

Eq. (18) may be written in matrix form as

$$[\mathrm{K}]\{d\}\text{-}\omega^2[\mathrm{M}]\{d\} = \{0\} \qquad (19)$$

where $K_{ij}=k_e(i^2j^2\pi^4/2L^3)b_ib_j + R_{ij}$, in which $R_{ij}=0$ if $i\neq j$, and $R_{ii}=EI(i\pi/L)^4\text{-}(i\pi/L)^2\, P$

The mass matrix [M] is diagonal. All of its diagonal elements are given by

$$M_{ii} = \rho A \qquad (20)$$

If $[K]\text{-}\omega^2[M]=[G]$, where $[G]$ is a dynamic stiffness matrix, then for non-trivial solution

$$\det[G] = 0 \qquad (21)$$

This is the frequency equation. For an initial imperfection in the form of a buckling mode $\sin(i\pi x/L)$, the stiffness matrix would also be diagonal. This leads to the following expression for the natural frequency

$$(\omega_i/\Omega_i)^2 = 1\text{-}(P/P_{ci}) + ½\, k_e'(b_i/r)^2 \qquad (22)$$

where Ω_i is the i^{th} natural frequency of a stress free straight beam and is given by $\Omega_i = (i\pi/L)^2 r(E/\rho)^{½}$, r is the radius of gyration about the relevant neutral axis, and k_e' is a non-dimensional effective end stiffness parameter given by $k_e'=k_eL/(EA)$.

The slope of the straight line representing the relationship between $(\omega_i/\Omega_i)^2 + (P/P_{ci})$ and $(b_i/r)^2$ is $k_e'/2$ and not $k_e'/(2i^4)$ as in (Ilanko and Dickinson 1986) which had an error.

For initial imperfections of other shapes, the coefficients a_i in Eq. (8) are given by

$$a_i = (2/L)\int_0^L v_o \sin(i\pi x/L)dx \qquad (23)$$

The coefficients b_i would be obtained from Eqs. (9a), (b). From Eq. (18) it may be noted that the stiffness matrix has non-zero off-diagonal terms which means that the simple relationship given by Eq. (22) would no longer be valid. It is however interesting to show how the relationship between the frequency and curvature, may be obtained for other types of imperfections. For longitudinally fully restrained (i.e., $ke'=1$), simply supported beams, the first frequency corresponding is also computed for the other types of imperfection. Labelling the sinusoidal imperfection as Type (1), the other types are:

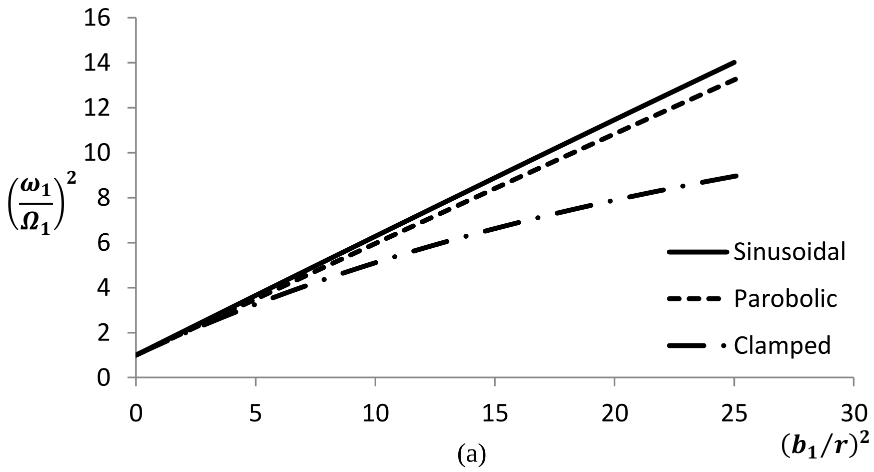
Type (2): Parabolic imperfection given by $v_0=4b_0(1\text{-}x/L)(x/L)$

Type (3): Approximate vibration mode of a clamped beam, $v_0=b_0\,(1\text{-}\cos\,(2\pi x/L))$

In both cases, b_0 is the magnitude of imperfection at the centre.

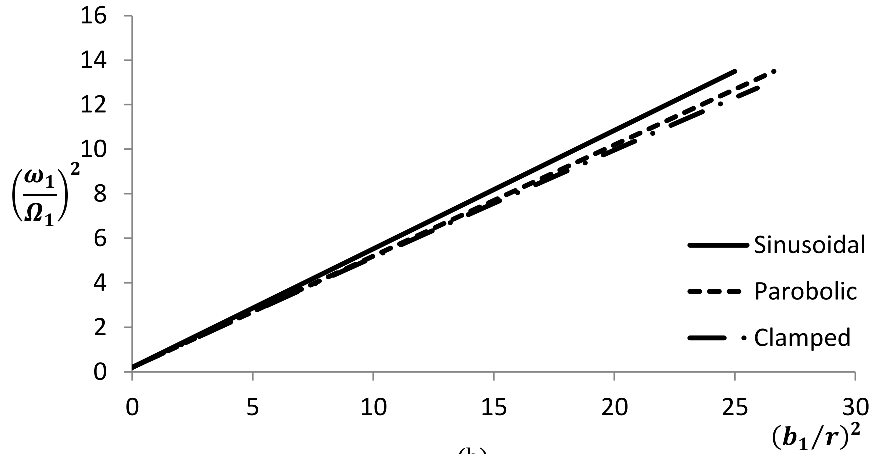
4. Results and discussion

The variation of the square of the fundamental natural frequency for simply supported beams with the square of the magnitude of imperfection at the centre for three cases and sinusoidal shape imperfection are presented in Fig. 2(a) and (b), which show the results when P is 0 and P is 80% of the first critical buckling load of a simply supported beam respectively. The numerical results are also presented in Table 1. While this relationship for the beam with parabolic imperfection is approximately linear, for Type (3) imperfection a highly non-linear relationship between the frequency squared and the square of central deflection is observed. Type (3) imperfection resembles the fundamental vibration mode of a clamped-clamped beam, and is an unusual form for a simply supported beam. The parabolic imperfection has a shape similar to the half-sine wave, the fundamental mode of simply supported straight beams. The deviation of the results from the linear relationship obtained for the sinusoidally curved beams is due to the contribution from other terms in the series for v_0. For beams that have small imperfections, the relationship may be regarded as approximately linear. It is also worth mentioning that in the cases of axially compressed beams, the initial imperfection increases with axial load, and tends to take the shape of the first buckling mode which is sinusoidal. Therefore the fundamental natural frequencies of axially compressed simply supported imperfect beams may be calculated using the simple linear relationship given by Eq. (22).



(a)

Fig. 2 The frequency–static deflection relationship for a simply supported beam (a) P=0, (b) P=0.8P_{c1}



(b)

Fig. 2 Continued

Table 1 The frequency-static deflection relationship for a simply supported beam for P=0, and P=0.8P_{c1}

			$(b_1/r)^2$					
			0	1	4	9	16	25
$\left(\frac{\omega_1}{\Omega_1}\right)^2$	P=0	Sinusoidal	1.0000	1.5321	3.1235	5.7579	9.4052	14.015
		Parabolic	1.0000	1.4996	2.9941	5.4691	8.8985	13.239
		Clamped	1.0000	1.4898	2.8441	4.7657	6.8979	8.9493
	P=0.8P_{c1}	Sinusoidal	0.2000	0.7325	2.3299	4.9912	8.7148	13.498
		Parabolic	0.2000	0.7000	2.1997	4.6984	8.1946	12.686
		Clamped	0.2000	0.6994	2.1911	4.6539	8.0484	12.309

Results for a beam that is fully laterally restrained against translation, and partially restrained against rotation are also obtained. The lateral displacement functions used in the Rayleigh-Ritz procedure are sinusoidal and are not valid permissible functions for a clamped-clamped beam. However, the frequencies of straight beams increase with rotational end stiffness parameters, and asymptotically approach the frequencies of a clamped-clamped beam for very high values of the stiffnesses. It should be noted here that with the sinusoidal functions a number of terms need to be used in order to obtain results close to those of a clamped-clamped beam. The results corresponding these nearly clamped beams for the three different types of imperfection are presented in Table 2 and Fig. 3(a) and (b) which show the results when P is 0 and P is 80% of the first critical buckling load of a clamped-clamped beam.

For the nearly clamped beam, the frequency squared - central imperfection squared relationship is approximately linear for beams having a sinusoidal and parabolic type imperfections but the clamped beam type. The slope of lines is much smaller than that of a simply supported beam case. As the axial load increase the lines move closer to each other and become approximately straight lines

Tables 3-8 and Figs. 4-9 show the variation of the sum of the square of the non-dimensional natural frequency and the axial load ratio with the square of the non-dimensional amplitude of deflection corresponding to the first sinusoidal shape (b_1/r). From this, it may be seen that the

Table 2 The frequency-static deflection relationship for a clamped beam for P=0, and P=0.8P_{c1}

			$(b_1/r)^2$					
			0	1	4	9	16	25
$\left(\frac{\omega_1}{\Omega_1}\right)^2$	P=0	Sinusoidal	1.0033	1.0979	1.3816	1.8542	2.5152	3.3637
		Parabolic	1.0033	1.0861	1.3341	1.7453	2.3164	3.0424
		Clamped	1.0033	1.1688	1.6520	2.4126	3.3869	4.4941
	P=0.8P_{c1}	Sinusoidal	0.2077	0.3015	0.5826	1.0505	1.7045	2.5431
		Parabolic	0.2077	0.2986	0.5707	1.0235	1.6554	2.4641
		Clamped	0.2077	0.3174	0.6462	1.1936	1.9587	2.9402

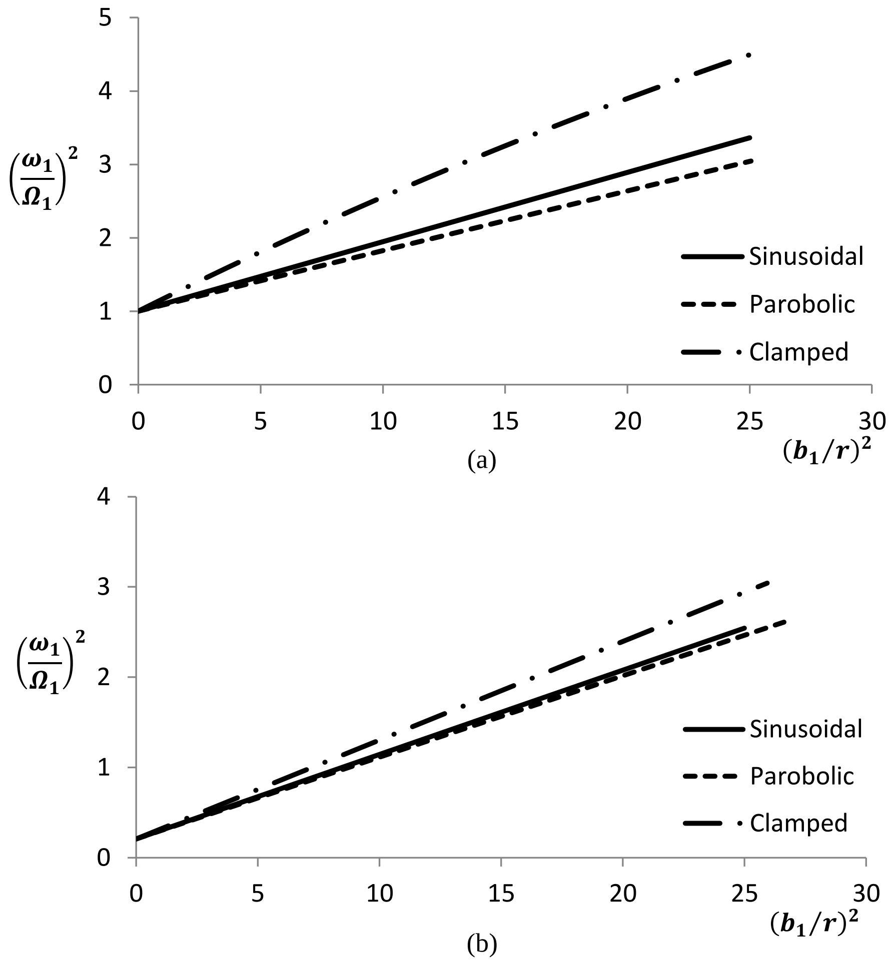


Fig. 3 The frequency-static deflection relationship for a clamped beam (a) P=0, (b) P=0.8P_{c1}

relationship between these two parameters is approximately linear. The linearity is exact for a simply-supported beam with a pure sinusoidal imperfection in the first natural mode but in other cases it is approximate.

Table 3 The variation of the sum of the square of the non-dimensional natural frequency and the axial load ratio with the square of the non-dimensional amplitude of deflection for the simply supported beam with the sinusoidal imperfection

	Initial imperfection at the centre					
			0.1r			
$(b_1/r)^2$	0.01	0.04	0.16	0.25	1	4
$(\omega_1/\Omega_1)^2+(P/P_{c1})$	1.0053	1.0213	1.0852	1.1331	1.5326	3.1302
			0.3r			
$(b_1/r)^2$	0.09	0.25	1	4	9	36
$(\omega_1/\Omega_1)^2+(P/P_{c1})$	1.0479	1.1331	1.5325	3.1300	5.7926	20.170
			0.5r			
$(b_1/r)^2$	0.25	1	4	6.25	25	100
$(\omega_1/\Omega_1)^2+(P/P_{c1})$	1.1331	1.5324	3.1297	4.3276	14.310	54.204

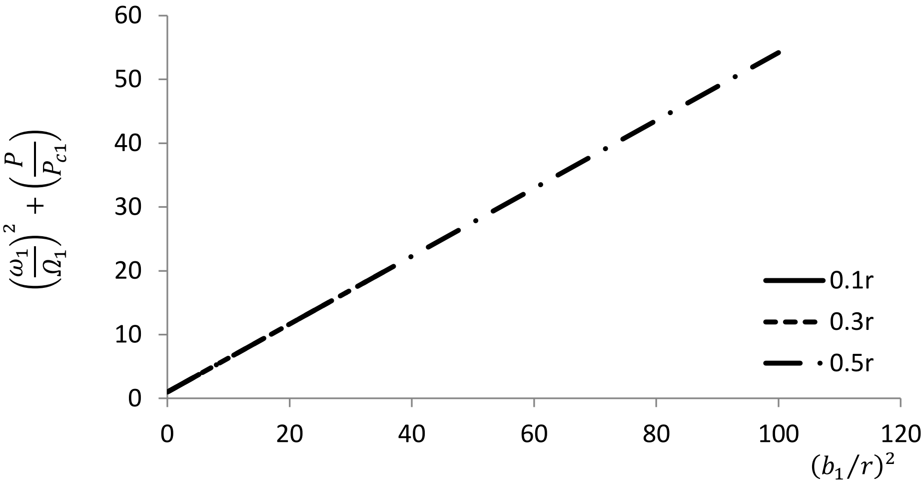


Fig. 4 The variation of the sum of the square of the non-dimensional natural frequency and the axial load ratio with the square of the non-dimensional amplitude of deflection for the simply supported beam with the sinusoidal imperfection

Table 4 The variation of the sum of the square of the non-dimensional natural frequency and the axial load ratio with the square of the non-dimensional amplitude of deflection for the simply supported beam with Type (2) imperfection

	Initial imperfection at the centre					
			0.1r			
$(b_1/r)^2$	0.0107	0.0252	0.0869	0.2663	1.0651	4.2605
$(\omega_1/\Omega_1)^2+(P/P_{c1})$	1.0053	1.0126	1.0435	1.1331	1.5326	3.1302
			0.3r			
$(b_1/r)^2$	0.1062	0.2663	1.0651	4.2605	9.5861	38.345
$(\omega_1/\Omega_1)^2+(P/P_{c1})$	1.0531	1.1331	1.5325	3.1300	5.7926	20.170
			0.5r			
$(b_1/r)^2$	0.2663	1.0651	4.2605	6.6570	26.628	106.51
$(\omega_1/\Omega_1)^2+(P/P_{c1})$	1.1331	1.5324	3.1297	4.3276	14.310	54.204

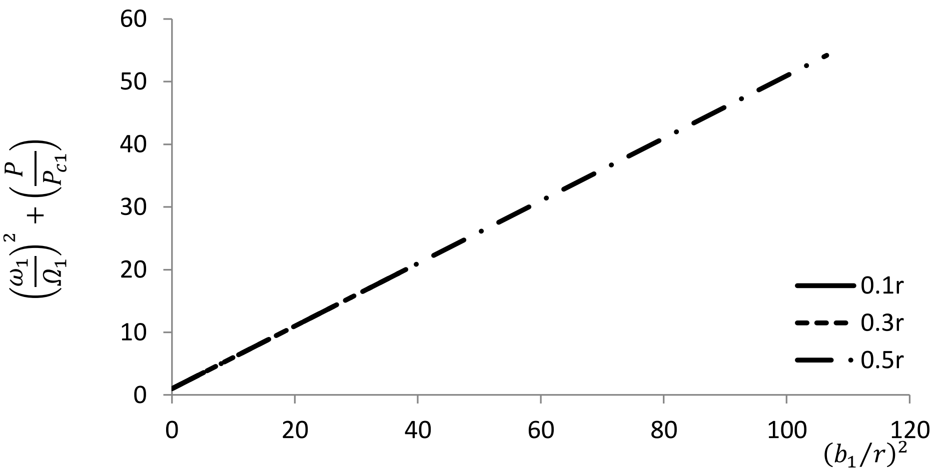


Fig. 5 The variation of the sum of the square of the non-dimensional natural frequency and the axial load ratio with the square of the non-dimensional amplitude of deflection for the simply supported beam with Type (2) imperfection

Table 5 The variation of the sum of the square of the non-dimensional natural frequency and the axial load ratio with the square of the non-dimensional amplitude of deflection for the simply supported beam with Type (3) imperfection

	Initial imperfection at the centre					
	0.1r					
$(b_1/r)^2$	0.0100	0.0450	0.1801	0.3202	0.7205	2.8820
$(\omega_1/\Omega_1)^2+(P/P_{c1})$	1.0050	1.0225	1.0900	1.1601	1.3602	2.4407
	0.3r					
$(b_1/r)^2$	0.1013	0.1535	0.2594	1.0375	6.4846	25.938
$(\omega_1/\Omega_1)^2+(P/P_{c1})$	1.0506	1.0766	1.1295	1.5179	4.2361	13.940
	0.5r					
$(b_1/r)^2$	0.2493	0.5004	1.1258	4.5032	18.013	72.051
$(\omega_1/\Omega_1)^2+(P/P_{c1})$	1.1242	1.2491	1.5603	3.2404	9.9551	36.668

Table 6 The variation of the sum of the square of the non-dimensional natural frequency and the axial load ratio with the square of the non-dimensional amplitude of deflection for the clamped beam with the sinusoidal imperfection.

	Initial imperfection at the centre					
	0.1r					
$(b_1/r)^2$	0.01	0.04	0.16	0.25	1	4
$(\omega_1/\Omega_1)^2+(P/P_{c1})$	1.0042	1.0145	1.0237	1.0312	1.0988	1.3774
	0.3r					
$(b_1/r)^2$	0.09	0.25	1	4	9	36
$(\omega_1/\Omega_1)^2+(P/P_{c1})$	1.0118	1.0340	1.1034	1.3811	1.8465	4.3498
	0.5r					
$(b_1/r)^2$	0.25	1	4	6.25	25	100
$(\omega_1/\Omega_1)^2+(P/P_{c1})$	1.0269	1.1049	1.3838	1.5932	3.3362	5.0859

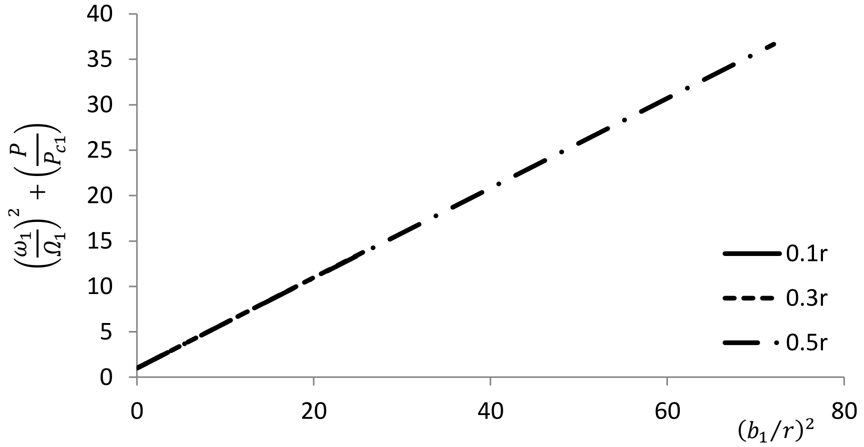


Fig. 6 The variation of the sum of the square of the non-dimensional natural frequency and the axial load ratio with the square of the non-dimensional amplitude of deflection for the simply supported beam with Type (3) imperfection

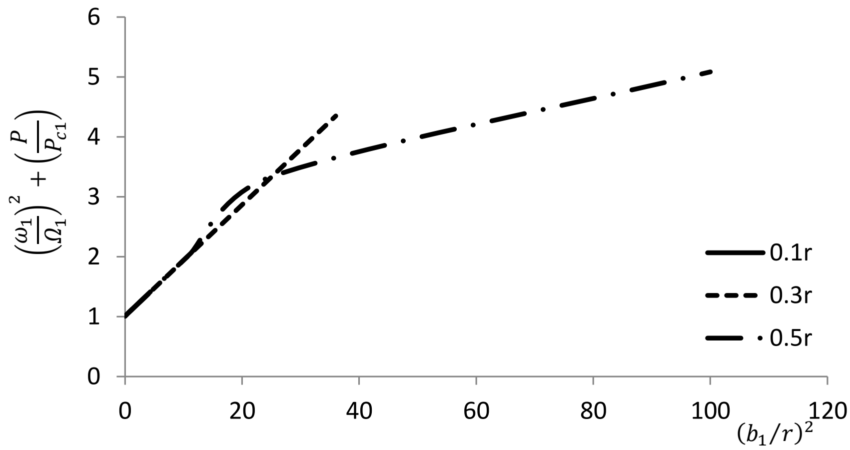


Fig. 7 The variation of the sum of the square of the non-dimensional natural frequency and the axial load ratio with the square of the non-dimensional amplitude of deflection for the clamped beam with the sinusoidal imperfection

Table 7 The variation of the sum of the square of the non-dimensional natural frequency and the axial load ratio with the square of the non-dimensional amplitude of deflection for the clamped beam with Type (2) imperfection

	Initial imperfection at the centre					
	$0.1r$					
$(b_1/r)^2$	0.0107	0.0426	0.1704	0.2663	1.0651	4.2605
$(\omega_1/\Omega_1)^2+(P/P_{c1})$	1.0041	1.0145	1.0241	1.0319	1.1032	1.3985
	$0.3r$					
$(b_1/r)^2$	0.0959	0.2663	1.0651	4.2605	9.5861	38.345
$(\omega_1/\Omega_1)^2+(P/P_{c1})$	1.0112	1.0334	1.1049	1.3959	1.8864	4.5329
	$0.5r$					
$(b_1/r)^2$	0.2663	1.0651	4.2605	6.6570	26.628	106.51
$(\omega_1/\Omega_1)^2+(P/P_{c1})$	1.0253	1.1039	1.3927	1.6115	3.4437	5.0859

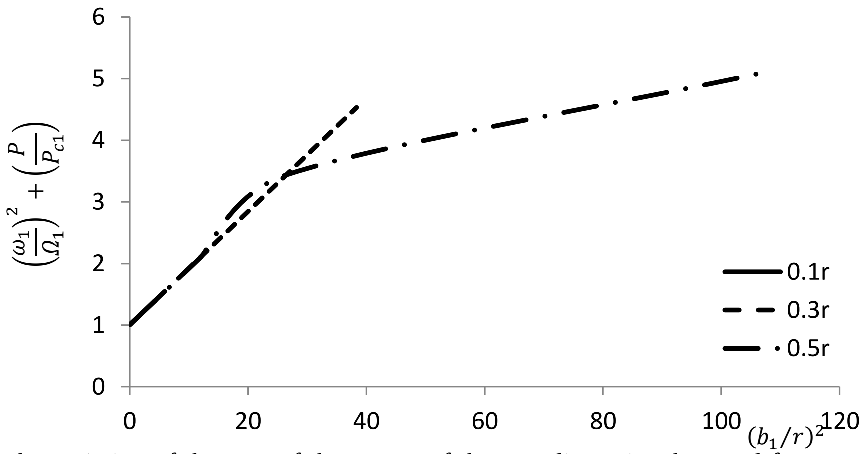


Fig. 8 The variation of the sum of the square of the non-dimensional natural frequency and the axial load ratio with the square of the non-dimensional amplitude of deflection for the clamped beam with Type (2) imperfection

Table 8 The variation of the sum of the square of the non-dimensional natural frequency and the axial load ratio with the square of the non-dimensional amplitude of deflection for the clamped beam with Type (3) imperfection.

	Initial imperfection at the centre					
			0.1r			
$(b_1/r)^2$	0.0100	0.0450	0.1801	0.3202	0.7205	2.8820
$(\omega_1/\Omega_1)^2+(P/P_{c1})$	1.0085	1.0161	1.0275	1.0404	1.0784	1.2847
			0.3r			
$(b_1/r)^2$	0.1013	0.1535	0.2594	1.0375	6.4846	25.938
$(\omega_1/\Omega_1)^2+(P/P_{c1})$	1.0235	1.0320	1.0451	1.1265	1.6642	3.5312
			0.5r			
$(b_1/r)^2$	0.2493	0.5004	1.1258	4.5032	18.013	72.051
$(\omega_1/\Omega_1)^2+(P/P_{c1})$	1.0460	1.0803	1.1512	1.5013	2.8359	5.0859

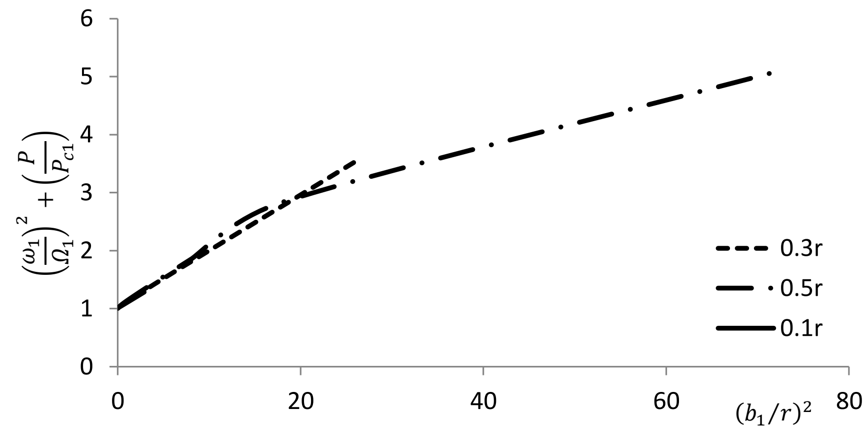


Fig. 9 The variation of the sum of the square of the non-dimensional natural frequency and the axial load ratio with the square of the non-dimensional amplitude of deflection for the clamped beam with Type (3) imperfection

For the case of a simply supported beam, the Rayleigh-Ritz procedure described here is more tedious than the Newtonian approach. However, as can be seen from the example presented here, it is more versatile since it permits inclusion of extra partial (lateral) restraints.

4. Conclusions

A Rayleigh-Ritz procedure for calculating the natural frequencies of slightly curved axially loaded beams has been presented. It has been shown that, if the longitudinal inertia is neglected, some of the Rayleigh-Ritz minimization equations are independent of the frequency. These equations can be used to formulate a relationship between the weighting coefficients associated with the lateral and longitudinal displacements. Substituting this relationship into the remaining minimization equations yields a condensed matrix equation in the standard form of an eigenvalue problem. The natural frequencies of the simply supported and partially clamped beams with three different shapes of imperfection are obtained using this method. The results indicate that for small imperfections, there exists an approximate linear relationship between the square of the fundamental natural frequency and the square of the central displacement.

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