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Forced vibration analysis of viscoelastic nanobeams embedded in an elastic medium

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Abstract. Forced vibration analysis of a simple supported viscoelastic nanobeam is studied based on modified couple stress theory (MCST). The nanobeam is excited by a transverse triangular force impulse modulated by a harmonic motion. The elastic medium is considered as Winkler-Pasternak elastic foundation.The damping effect is considered by using the Kelvin–Voigt viscoelastic model. The inclusion of an additional material parameter enables the new beam model to capture the size effect. The new non-classical beam model reduces to the classical beam model when the length scale parameter is set to zero. The considered problem is investigated within the Timoshenko beam theory by using finite element method. The effects of the transverse shear deformation and rotary inertia are included according to the Timoshenko beam theory. The obtained system of differential equations is reduced to a linear algebraic equation system and solved in the time domain by using Newmark average acceleration method. Numerical results are presented to investigate the influences the material length scale parameter, the parameter of the elastic medium and aspect ratio on the dynamic response of the nanobeam. Also, the difference between the classical beam theory (CBT) and modified couple stress theory is investigated for forced vibration responses of nanobeams.

Keywords: nanobeam; modified couple stress theory; forced vibration; winkler-pasternak foundation

1. Introduction

With the great advances in technology in recent years, micro and nano structures have found many applications. In these structures, micro beams and micro tubes are widely used in micro- and nano electromechanical systems (MEMS and NEMS) such as sensors (Zook, Burns *et al.* 1992, Pei, Tian *et al*. 2004), microactuators (Senturia 1998, Rezazadeh, Tahmasebi *et al*. 2006), atomic force microscopes, micro-resonators. In investigation of micro and nano structures, the classical continuum mechanics which is scale independent theories, are not capable of explanation of the size-dependent behaviors. Nonclassical continuum theories such as higher order gradient theories and the couple stress theory are capable of explanation of the size dependent behaviors which occur in micro-scale structures.

At the present time, the experimental investigations of the micro materials are still a challenge because of difficulties confronted in the micro scale. Therefore, mechanical theories and atomistic

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simulations have been used for micro structural analysis. The process of the atomistic simulations is very difficult and takes much time. So, continuum theory is the most preferred method for the analysis of the micro and nano structures. Classical continuum mechanics does not contain the size effect, because of its scale-free character. The nonlocal continuum theory initiated by Eringen (1972) which has been widely used to mechanical behavior of nano-micro structures.

The size effect plays an important role on the mechanical behavior of microstructures at the micrometer scale that the classic theory has failed to consider when the size reduces from macro to nano (Toupin 1962, Mindlin 1962, Mindlin 1963, Fleck and Hutchinson 1993, Yang, Chong *et al.* 2002, Lam, Yang *et al.* 2003). Therefore, higher-order theories modified couple stress theory and modified strain gradient are used in the mechanical model of the nano-micro structures (Yang, Chong *et al.* 2002, Lam, Yang *et al.* 2003).

The determination of the micro-structural material length scale parameters is very difficult experimentally. So, Yang, Chong *et al.* (2002) proposed the modified couple stress theory (MCST) in which the strain energy has been shown to be a quadratic function of the strain tensor and the symmetric part of the curvature tensor, and only one length scale parameter is included. After this, the modified couple stress and the strain gradient elasticity theories have been widely applied to static and dynamic analysis of beams (Park and Gao 2006, Ma *et al.* 2008, Kong *et al.* 2008, Asghari, Ahmadian *et al.* 2010, Wang 2010, Şimşek 2010, Kahrobaiyan *et al.* 2010, Xia *et al.* 2010, Ke, Wang *et al.* 2011, Kahrobaiyan, Asghari *et al.* 2011, Akgöz and Civalek 2012a, 2014a, b, 2016, Movahedian 2012, Ansari, Ahmadian *et al.* 2012a,b, Şimşek, Kocatürk *et al.* 2013, Wang, Xu *et al.* 2013, Kocatürk and Akbaş 2013, Tounsi, Benguediab *et al.* 2013, Kong 2013, Ghayesh, Amabili *et al.* 2013, Daneshmehr, Abadi *et al.* 2013, Akgöz and Civalek 2013, Şimşek and Reddy 2013, Bayat, Pakar *et al.* 2013, Bahraini, Eghtesad *et al.* 2014, Afkhami and Farid 2014, Besseghier, Heireche *et al.* 2015, Benguediab, Tounsi *et al.* 2014, Mohammadimehr, Mohandes *et al.* 2014).

More recently, Darijani and Mohammadabadi (2014) proposed a new deformation beam theory for static and dynamic analysis of microbeams which includes unknown functions takes into account shear deformation and satisfies both of shear and couple-free conditions on the upper and lower surfaces of the beam based on a modified couple stress theory. Tang, Ni *et al.* (2014) analyzed a theoretical model for flexural vibrations of microbeams in flow with clamped-clamped ends based on a modified couple stress theory. Sedighi, Changizian *et al.* (2014) investigated the dynamic pull-in instability of vibrating micro-beams undergoing large deflection under electrostatically actuation. Faraokhi and Ghayesh (2015) studied the three-dimensional motion characteristics of perfect and imperfect Timoshenko microbeams under mechanical and thermal forces based on the modified couple stress theory. Kural and Özkaya (2015) studied the vibration of a micro beam with conveying fluid and resting on an elastic foundation. Farokhi and Ghayesh (2015a) investigated the three-dimensional motion characteristics of perfect and imperfect Timoshenko microbeams under mechanical and thermal forces. Ansari, Ashrafi *et al.* (2015) studied an exact solution of vibrations of postbuckled microscale beams based on the modified couple stress theory. Al-Basyouni, Tounsi *et al.* (2015) studied bending and dynamic behaviors of functionally graded micro beams with a novel unified beam formulation and a MCST. Zamanian, Rezaei *et al.* (2015) investigated the mechanics behavior of piezoelectrically actuated microbeams on the discretization methods. Dai, Wang *et al.* (2015) developed a new nonlinear theoretical model for cantilevered microbeams and explore the nonlinear dynamics based on the modified couple stress theory, taking into account one single material length scale parameter. Farokhi and Ghayesh (2015b) investigated the nonlinear dynamics of microarches with internal modal interactions. Chaht, Kaci *et al.* (2015) studied bending and buckling of size-dependent functionally graded nanobeams including the thickness stretching effect. The mechanical responses of microbeams studied within shear deformation and strain gradient theories by Akgöz and Civalek (2014, 2015a, b), Zemri, Houari *et al*. (2015), Bounouara, Benrahou *et al*. (2016), Shafiei, Mousavi *et al.* (2016). Aissani, Bouiadjra *et al.* (2015) presented a new nonlocal hyperbolic shear deformation beam theory for the static, buckling and vibration of nanobeams embedded in an elastic medium. Bağdatlı (2015) studied nonlinear transverse vibration of tensioned Euler-Bernoulli nanobeams. Akbaş (2016) studied the analytical solution of cracked microbeams using MCST. Ahouel, Houari *et al.* (2016) investigated the bending, buckling, and vibration of functionally graded nanobeams using the nonlocal differential constitutive relations of Eringen. Ebrahimi and Shafiei (2016) investigated size dependent vibration of a rotating functionally graded nanobeam with Timoshenko beam theory based on Eringen's nonlocal theory.

In this study, the forced vibration response of a simple supported nanobeam embedded in an elastic medium is studied under the effect of a force impulse based on the MCST theory within the Timoshenko beam theory by using finite element method. The effects of the transverse shear deformation and rotary inertia are included according to the Timoshenko beam theory. The elastic medium is considered as Winkler-Pasternak elastic foundation. The Kelvin–Voigt viscoelastic model is used for the material of the nanobeam. In the dynamic solution of the problem, the Newmark average acceleration method is used. The effect of the material length scale parameter, the parameter of the elastic medium and aspect ratio on the forced vibration responses of the nanobeam are investigated in both the classical beam theory (CBT) and MCST.

2. Theory and formulations

Consider a simple supported circular nanobeam of length *L*, diameter *D*, as shown in Fig. 1. The nanobeam is excited by a transverse triangular force (P) impulse modulated by a harmonic motion at the midpoint of the beam and resting on Winkler-Pasternak foundation with spring constant k_w and k_p, as as seen from Fig. 1. When the Pasternak foundation spring constant $k_p=0$, the foundation model reduces to Winkler type.

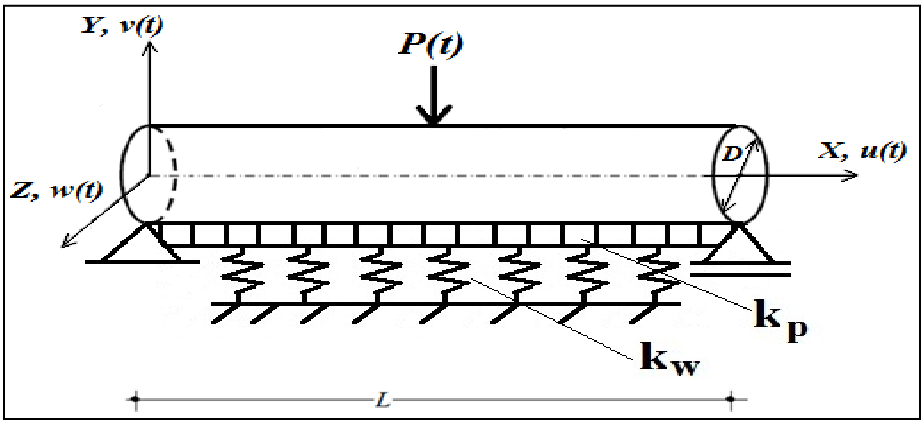

Fig. 1 A simple supported nanobeam resting on Winkler-Pasternak foundation subjected to a force impulse

2.1 The modified couple stress theory

The modified couple stress theory was proposed by Yang, Chong *et al*. (2002). Based on this theory, the strain energy density for a linear elastic material which is a function of both strain tensor and curvature tensor is introduced for the modified couple stress theory

$$U = \int_V (\boldsymbol{\sigma} : \boldsymbol{\varepsilon} + \boldsymbol{m} : \boldsymbol{\chi})\, dV \tag{1}$$

where $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{\varepsilon}$ is the strain tensor, $\boldsymbol{m}$ is the deviatoric part of the couple stress tensor, $\boldsymbol{\chi}$ is the symmetric curvature tensor, defined by

$$\boldsymbol{\sigma} = \lambda\, tr(\boldsymbol{\varepsilon}) I + 2\mu \boldsymbol{\varepsilon} \tag{2}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T] \tag{3}$$

$$\boldsymbol{m} = 2l^2 \mu\, \boldsymbol{\chi} \tag{4}$$

$$\boldsymbol{\chi} = \frac{1}{2}[\nabla \boldsymbol{\theta} + (\nabla \boldsymbol{\theta})^T] \tag{5}$$

where λ and μ are Lame’s constants, l is a material length scale parameter which is regarded as a material property characterizing the effect of couple stress, $\boldsymbol{u}$ is the displacement vector and $\boldsymbol{\theta}$ is the rotation vector, given by

$$\boldsymbol{\theta} = \frac{1}{2} \text{curl}\, \boldsymbol{u} \tag{6}$$

The parameters λ and μ in the constitutive equation are given by

$$\lambda = \frac{E\ \nu}{(1+\nu)(1-2\nu)}\ , \quad \mu = \frac{E}{2(1+\nu)} \tag{7}$$

where E is the modulus of elasticity and ν is the Poisson's ratio.

2.2 Governing equations of the problem

According to the coordinate system *(X,Y,Z)* shown in Fig. 1, based on Timoshenko beam theory, the axial and the transverse displacement field are expressed as

$$u(X,Y,t) = u_0(X,t) - Y\emptyset(X,t) \tag{8}$$

$$v(X,Y,t) = v_0(X,t) \tag{9}$$

$$w(X,Y,t) = 0 \tag{10}$$

where u, v, w are x, y and z components of the displacements, respectively. $\emptyset$ is the total bending rotation of the cross-sections at any point on the neutral axis. Also, u_0 and v_0 are the axial and the transverse displacements in the mid-plane, t indicates time.

Because the transversal surfaces of the beam is free of stress, then

$$\sigma_{zz} = \sigma_{yy} = 0 \tag{11}$$

By using Eqs. (3), (8) and (9) and strain- displacement relation can be obtained

$$\varepsilon_{xx} = \frac{\partial u}{\partial X} = \frac{\partial u_0(X,t)}{\partial X} - Y\frac{\partial \emptyset(x,t)}{\partial X} \tag{12a}$$

$$\varepsilon_{xz} = \varepsilon_{yz} = 0 \tag{12b}$$

$$2\varepsilon_{xy} = \gamma_{xy} = \frac{\partial v}{\partial \mathrm{x}} - \emptyset(x,t) \tag{12c}$$

By using Eqs. (6), (8), (9) and (10),

$$\theta_z = \frac{1}{2}\left(\frac{\partial v(X,t)}{\partial X} + \emptyset(x,t)\right), \quad \theta_x = \theta_y = 0 \tag{13}$$

Substituting Eq, (13) into Eq. (5), the curvature tensor $\boldsymbol{\chi}$ can be obtained as follows

$$\chi_{xz} = \frac{1}{4}\left(\frac{\partial^2 v(X,t)}{\partial X^2} + \frac{\partial \emptyset(X,t)}{\partial X}\right), \quad \chi_{xx} = \chi_{xy} = \chi_{yy} = \chi_{yz} = \chi_{zz} = 0 \tag{14}$$

In the damping effect, the Kelvin–Voigt viscoelastic model is used for the material. The constitutive relations for the Kelvin– Voigt viscoelastic model between the stresses and strains become

$$\sigma_{xx} = E(\varepsilon_{xx} + \eta_1 \dot{\varepsilon}_{xx}) \tag{15a}$$

$$\sigma_{xy} = k_s\, \mu(\gamma_{xy} + \eta_2 \dot{\gamma}_{xy}) \tag{15b}$$

$$\boldsymbol{m} = 2l^2\mu\, (\boldsymbol{\chi} + \eta_3 \dot{\boldsymbol{\chi}}) \tag{15c}$$

where E, σ_{xx}, $\varepsilon_{xx}, \sigma_{xy}, \gamma_{xy}, \boldsymbol{m}, \boldsymbol{\chi}$, $\dot{\varepsilon}_{xx}$, $\dot{\gamma}_{xy}$ and $\dot{\boldsymbol{\chi}}$ indicate Young's modulus, normal stresses, normal strains in the X direction, shear stresses, shear strains, the couple stress tensor, the symmetric curvature tensor, the time derivatives of the normal strains, time derivatives of the shear strains and the time derivatives of the curvature tensor, respectively. μ is shear modulus which is defined by Eq. (7). Also, η_1, η_2 and η_3 indicate the damping ratios of the viscoelastic nanobeam in bending, shearing and couple stress, respectively, as follows

$$\eta_1 = \frac{c}{E},\ \eta_2 = \frac{c}{\mu},\ \eta_3 = \frac{c}{2l^2\mu} \tag{16}$$

where c is the damping coefficient.

Substituting Eqs. (14) and (12) into Eq. (15), the stresses can be obtained as follows

$$\sigma_{xx} = E\left(\left(\frac{\partial u_0(X,t)}{\partial X} - Y\frac{\partial \emptyset(x,t)}{\partial X}\right) + \eta_1 \frac{\partial}{\partial t}\left(\frac{\partial u_0(X,t)}{\partial X} - Y\frac{\partial \emptyset(x,t)}{\partial X}\right)\right) \tag{17a}$$

$$\sigma_{xy} = k_s\, \mu\left(\left(\frac{\partial v}{\partial \mathrm{x}} - \emptyset(x,t)\right) + \eta_2 \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial \mathrm{x}} - \emptyset(x,t)\right)\right) \tag{17b}$$

$$m_{xz} = \frac{1}{2}l^2\mu\left(\left(\frac{\partial^2 v(X,t)}{\partial X^2} + \frac{\partial \emptyset(X,t)}{\partial X}\right) + \eta_3 \frac{\partial}{\partial t}\left(\frac{\partial^2 v(X,t)}{\partial X^2} + \frac{\partial \emptyset(X,t)}{\partial X}\right)\right) \tag{17c}$$

$$m_{xx} = m_{xy} = m_{yy} = m_{yz} = m_{zz} = 0 \tag{17d}$$

when the total bending rotation is $\emptyset = {}^{\partial \mathrm{v}}\!/_{\partial \mathrm{x}}$, the beam model reduces to Euler-Bernoulli beam model.

Based on Timoshenko beam theory, the elastic strain energy (U_i) of the nanobeam is expressed as

$$U_i = \frac{1}{2}\int_0^L \int_A \left(\sigma_{ij}\, \varepsilon_{ij} + m_{ij}\, \chi_{ij}\right) dA\, dX \tag{18}$$

By substituting Eqs. (12), (14) and (17) into Eq. (18), elastic strain energy (U_i) can be rewritten as follows

$$U_i = \frac{1}{2}\int_0^L \left[EA\left(\frac{\partial u_0(X,t)}{\partial X}\right)^2 + EI\left(\frac{\partial \emptyset(x,t)}{\partial X}\right)^2 + k_s\mu A\left(\frac{\partial v}{\partial \mathrm{x}} - \emptyset(x,t)\right)^2 + \frac{1}{8}l^2\mu A\left(\frac{\partial^2 v(X,t)}{\partial X^2} + \frac{\partial \emptyset(X,t)}{\partial X}\right)^2 + k_w(v(X,t))^2 + k_p(\frac{\partial v(X,t)}{\partial \mathrm{x}})^2\right] dX \tag{19}$$

where A is the area of the cross section, and I is the moment of inertia.

The kinetic energy (T) of the nanobeam is expressed as follows

$$T = \frac{1}{2}\int_0^L \int_A \rho \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] dA\, dX \tag{20}$$

By substituting Eqs. (8), (9) and (10) into Eq. (20), the kinetic energy (T) can be rewritten as follows

$$T = \frac{1}{2}\int_0^L \left[\rho A\left(\frac{\partial u_0}{\partial t}\right)^2 + \rho A\left(\frac{\partial v_0}{\partial t}\right)^2 + \rho I\left(\frac{\partial \emptyset}{\partial t}\right)^2\right] dX \tag{21}$$

where ρ is the mass density of the nanobeam.

The dissipation function of the nanobeam at any instant t is

$$R = \frac{1}{2}\int_0^L \left[\eta_1 EA\left(\frac{\partial \dot{u}_0(X,t)}{\partial X}\right)^2 + \eta_1 EI\left(\frac{\partial \dot{\emptyset}(x,t)}{\partial X}\right)^2 + \eta_2 k_s\mu A\left(\frac{\partial \dot{v}}{\partial \mathrm{x}} - \dot{\emptyset}(x,t)\right)^2 + \eta_3\frac{1}{8}l^2\mu A\left(\frac{\partial^2 \dot{v}(X,t)}{\partial X^2} + \frac{\partial \dot{\emptyset}(X,t)}{\partial X}\right)^2\right] dX \tag{22}$$

where "." indicates the time derivative. The potential energy of the external load can be written as

$$U_e = -\int_0^L [P(x,t)\; v(x,t)]\, dX \tag{23}$$

The Lagrangian functional of the problem is given as follows

$$I = T - (U_\mathrm{i} + U_\mathrm{e}) \tag{24}$$

Total nodal displacements q which is written for a two-node beam element, each node has three degrees of freedom, shown in Fig. 2 are defined as follows

$$\{q(t)\}_\mathrm{e} = \left[u_i^{(e)}(t), v_i^{(e)}(t),\ \emptyset_i^{(e)}(t), u_j^{(e)}(t), v_j^{(e)}(t),\ \emptyset_j^{(e)}(t)\ \right]^T \tag{25}$$

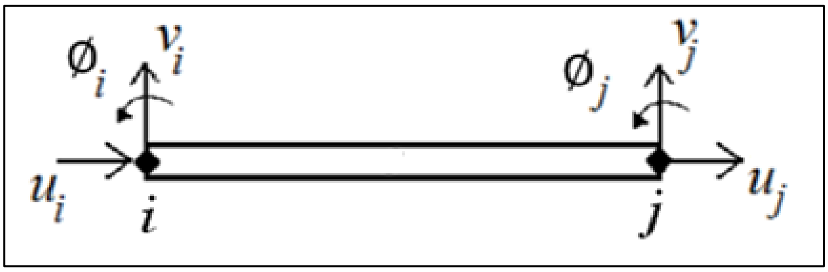

Fig. 2 A two-node finite element

The displacement field of the finite element is expressed in terms of nodal displacements as follows

$$u^{(\mathrm{e})}(X,\mathrm{t}) = \varphi_1^{(U)}(X)\, u_\mathrm{i}(t) + \varphi_2^{(U)}(X)\, u_\mathrm{j}(t)$$
$$= \left[\varphi^{(U)}\right]\begin{Bmatrix} u_\mathrm{i} \\ u_\mathrm{j} \end{Bmatrix} = \left[\varphi^{(U)}\right]\{\mathrm{q}\}_U \tag{26}$$

$$v^{(\mathrm{e})}(X,\mathrm{t}) = \varphi_1^{(V)}(X)\, v_\mathrm{i}(t) + \varphi_2^{(V)}(X)\, \emptyset_\mathrm{i}(t) + \varphi_3^{(V)}(X)\, v_\mathrm{j}(t) + \varphi_4^{(V)}(X)\, \emptyset_\mathrm{j}(t)$$
$$\left[\varphi^{(V)}\right]\begin{Bmatrix} v_\mathrm{i} \\ \emptyset_\mathrm{i} \\ v_\mathrm{j} \\ \emptyset_\mathrm{j} \end{Bmatrix} = \left[\varphi^{(V)}\right]\{q\}_V \tag{27}$$

$$\emptyset^{(\mathrm{e})}(X,\mathrm{t}) = \varphi_1^{(\emptyset)}(X)\, v_\mathrm{i}(t) + \varphi_2^{(\emptyset)}(X)\, \emptyset_\mathrm{i}(t) + \varphi_3^{(\emptyset)}(X)\, v_\mathrm{j}(t) + \varphi_4^{(\emptyset)}(X)\, \emptyset_\mathrm{j}(t)$$
$$\left[\varphi^{(\emptyset)}\right]\begin{Bmatrix} v_\mathrm{i} \\ \emptyset_\mathrm{i} \\ v_\mathrm{j} \\ \emptyset_\mathrm{j} \end{Bmatrix} = \left[\varphi^{(\emptyset)}\right]\{q\}_\emptyset \tag{28}$$

where u_i, v_i and $\emptyset_\mathrm{i}$ are axial displacements, transverse displacements and rotations at the two end nodes of the beam element, respectively. $\varphi_i^{(U)}$, $\varphi_i^{(V)}$ and $\varphi_i^{(\emptyset)}$ are shape functions for axial, transverse and rotation degrees of freedom, respectively, and are given as Chakraborty, Mahapatra *et al*. (2002).

The shape functions for axial degrees of freedom are

$$\varphi^{(U)}(\mathrm{X}) = \left[\varphi_1^{(U)}(\mathrm{X})\ \ \varphi_2^{(U)}(\mathrm{X})\right]^T \tag{29}$$

where

$$\varphi_1^{(U)}(\mathrm{X}) = \left(-\frac{X}{L_\mathrm{e}} + 1\right) \tag{30a}$$

$$\varphi_2^{(U)}(\mathrm{X}) = \left(\frac{X}{L_\mathrm{e}}\right) \tag{30b}$$

The shape functions for transverse degrees of freedom are

$$\varphi^{(V)}(\mathrm{X}) = \left[\varphi_1^{(V)}(\mathrm{X})\ \ \varphi_2^{(V)}(\mathrm{X})\ \ \varphi_3^{(V)}(\mathrm{X})\ \ \varphi_4^{(V)}(\mathrm{X})\right]^T \tag{31}$$

where

$$\varphi_1^{(V)}(\mathrm{X}) = \left(1 - \frac{12X}{L_\mathrm{e}\left(12+\frac{\mu A}{EI}L_\mathrm{e}^2\right)} - \frac{3\left(\frac{\mu A}{EI}\right)X^2}{\left(12+\frac{\mu A}{EI}L_\mathrm{e}^2\right)} + \frac{2(\mu A/EI)X^3}{L_\mathrm{e}\left(12+\frac{\mu A}{EI}L_\mathrm{e}^2\right)}\right) \tag{32a}$$

$$\varphi_2^{(V)}(\mathrm{X})=\left(\frac{\left(6+\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)X}{\left(12+\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)}-\frac{\left(6+2\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)X^2}{L_{\mathrm{e}}\left(12+\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)}+\frac{\mu A/EI}{6}\left(1-\frac{\left(6L_{\mathrm{e}}+\frac{\mu A}{EI}L_{\mathrm{e}}^3\right)}{L_{\mathrm{e}}\left(12+\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)}\right)X^3\right) \tag{32b}$$

$$\varphi_3^{(V)}(\mathrm{X})=\left(\frac{12X}{L_{\mathrm{e}}\left(12+\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)}+\frac{3\left(\frac{\mu A}{EI}\right)X^2}{\left(12+\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)}-\frac{2(\mu A/EI)X^3}{L_{\mathrm{e}}\left(12+\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)}\right) \tag{32c}$$

$$\varphi_4^{(V)}(\mathrm{X})=\left(-\frac{6X}{\left(12+\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)}-\frac{\left(\frac{\mu A}{EI}L_{\mathrm{e}}^2-6\right)X^2}{\left(12+\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)}+\frac{(\mu A/EI)X^3}{\left(12+\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)}\right) \tag{32d}$$

The shape functions for rotation degrees of freedom are

$$\varphi^{(\emptyset)}(\mathrm{X})=\left[\varphi_1^{(\emptyset)}(\mathrm{X})\ \ \varphi_2^{(\emptyset)}(\mathrm{X})\ \ \varphi_3^{(\emptyset)}(\mathrm{X})\ \ \varphi_4^{(\emptyset)}(\mathrm{X})\right]^T \tag{33}$$

where

$$\varphi_1^{(\emptyset)}(\mathrm{X})=\left(-\frac{6(\mu A/EI)X}{\left(12+\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)}+\frac{6(\mu A/EI)X^2}{L_{\mathrm{e}}\left(12+\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)}\right) \tag{34a}$$

$$\varphi_2^{(\emptyset)}(\mathrm{X})=\left(1-\frac{2\left(2\frac{\mu A}{EI}L_{\mathrm{e}}^2+6\right)X}{L_{\mathrm{e}}\left(12+\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)}+\frac{GA/EI}{2}\left(1-\frac{\left(6L_{\mathrm{e}}+\frac{\mu A}{EI}L_{\mathrm{e}}^3\right)}{L_{\mathrm{e}}\left(12+\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)}\right)X^2\right) \tag{34b}$$

$$\varphi_3^{(\emptyset)}(\mathrm{X})=\left(\frac{6\left(\frac{\mu A}{EI}\right)X}{\left(12+\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)}-\frac{6\left(\frac{\mu A}{EI}\right)X^2}{L_{\mathrm{e}}\left(12+\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)}\right) \tag{34c}$$

$$\varphi_4^{(\emptyset)}(\mathrm{X})=\left(-\frac{2\left(\frac{\mu A}{EI}L_{\mathrm{e}}^2-6\right)X}{L_{\mathrm{e}}\left(12+\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)}+\frac{3(\mu A/EI)X^2}{\left(12+\frac{\mu A}{EI}L_{\mathrm{e}}^2\right)}\right) \tag{34d}$$

where L_{e} indicates the length of the finite beam element.

By substituting Eqs. (26), (27) and (28) into Eqs. (19), (21) and (22), energy functions can be rewritten as follows

$$U_i=\frac{1}{2}\int_0^L\left[EA\left(\frac{\partial\varphi^{(U)}}{\partial X}\{q\}_U\right)^2+EI\left(\frac{\partial\varphi^{(\emptyset)}}{\partial X}\{q\}_\emptyset\right)^2+k_s\mu A\left(\frac{\partial\varphi^{(V)}}{\partial X}\{q\}_V-\varphi^{(\emptyset)}\{q\}_\emptyset\right)^2+\frac{1}{8}l^2\mu A\left(\frac{\partial^2\varphi^{(V)}}{\partial X^2}\{q\}_V+\frac{\partial\varphi^{(\emptyset)}}{\partial X}\{q\}_\emptyset\right)^2+k_w(v(X,t))^2+k_w\left(\frac{\partial v(X,t)}{\partial \mathrm{x}}\right)^2\right]dX \tag{35}$$

$$T=\frac{1}{2}\int_0^L\left[\rho A\left(\frac{\partial\left(\varphi^{(U)}\{q\}_U\right)}{\partial t}\right)^2+\rho A\left(\frac{\partial\left(\varphi^{(V)}\{q\}_V\right)}{\partial t}\right)^2+\rho I\left(\frac{\partial\left(\varphi^{(\emptyset)}\{q\}_\emptyset\right)}{\partial t}\right)^2\right]dX \tag{36}$$

$$R=\frac{1}{2}\int_0^L\left[\eta_1 EA\left(\frac{\partial}{\partial X}\left(\frac{\partial\left(\varphi^{(U)}\{q\}_U\right)}{\partial t}\right)\right)^2+\eta_1 EI\left(\frac{\partial}{\partial X}\left(\frac{\partial\left(\varphi^{(\emptyset)}\{q\}_\emptyset\right)}{\partial t}\right)\right)^2+\eta_2 k_s\mu A\left(\frac{\partial}{\partial X}\left(\frac{\partial\left(\varphi^{(V)}\{q\}_V\right)}{\partial t}\right)-\left(\frac{\partial\left(\varphi^{(\emptyset)}\{q\}_\emptyset\right)}{\partial t}\right)\right)^2+\eta_3\frac{1}{8}l^2\mu A\left(\frac{\partial^2}{\partial X^2}\left(\frac{\partial\varphi^{(V)}}{\partial t}\{q\}_V\right)+\frac{\partial}{\partial X}\left(\frac{\partial\varphi^{(\emptyset)}}{\partial t}\{q\}_\emptyset\right)\right)^2\right]dX \tag{37}$$

After substituting Eqs. (26), (27) and (28) into Eq. (24) and then using the Lagrange's equations gives the following equation

$$\frac{\partial I}{\partial q_k^{(e)}} - \frac{\partial}{\partial t}\frac{\partial I}{\partial \dot{q}_k^{(e)}} + Q_{D_k} = 0, \qquad \text{k=1,2,3,4,5,6} \tag{38}$$

where

$$Q_{D_k} = -\frac{\partial \mathrm{R}}{\partial \dot{q}_k^{(e)}} \tag{39}$$

Q_{D_k} is the generalized damping load which can be obtained from the dissipation function by differentiating R with respect to $\dot{q}_k^{(e)}$. Where $\dot{q}_k^{(e)}$ indicates the time derivatives of nodal displacements q.

The Lagrange's equations yield the system of equations of motion for the finite element and by use of usual assemblage procedure the following system of equations of motion for the whole system can be obtained as follows

$$[K]\{q(t)\} + [D]\{\dot{q}(t)\} + [M]\{\ddot{q}(t)\} = \{F(t)\} \tag{40}$$

where, $[K]$ is the stiffness matrix, $[D]$ is the damping matrix, $[M]$ is mass matrix and $\{F(t)\}$ is the load vector.

The components of the stiffness matrix $[K]$:

The stiffness matrix $[K]$ can be expressed as a sum of three submatrices as shown below

$$[K] = [K_b] + [K_w] + [K_p] \tag{41}$$

where

$$[K_b] = \begin{bmatrix} [K_b^U] & 0 & 0 \\ 0 & [K_b^V] & [K_b^{V\emptyset}] \\ 0 & [K_b^{\emptyset V}] & [K_b^{\emptyset}] \end{bmatrix} \tag{42a}$$

$$[K_b^U] = \int_0^{L_\mathrm{e}} EA \left[\frac{\partial \varphi^{(U)}}{\partial X}\right]^T \left[\frac{\partial \varphi^{(U)}}{\partial X}\right] dX \tag{42b}$$

$$[K_b^V] = \int_0^{L_\mathrm{e}} \left(k_s \mu A \left[\frac{\partial \varphi^{(V)}}{\partial X}\right]^T \left[\frac{\partial \varphi^{(V)}}{\partial X}\right] + \frac{1}{8} l^2 \mu A \left[\frac{\partial^2 \varphi^{(V)}}{\partial X^2}\right]^T \left[\frac{\partial^2 \varphi^{(V)}}{\partial X^2}\right] \right) dX \tag{42c}$$

$$[K_b^{V\emptyset}] = [K_b^{\emptyset V}]^\mathrm{T} = \int_0^{L_\mathrm{e}} \left(-k_s \mu A \left[\frac{\partial \varphi^{(V)}}{\partial X}\right]^T [\varphi^{(\emptyset)}] + \frac{1}{8} l^2 \mu A \left[\frac{\partial^2 \varphi^{(V)}}{\partial X^2}\right]^T \left[\frac{\partial \varphi^{(\emptyset)}}{\partial X}\right] \right) dX \tag{42d}$$

$$[K_b^{\emptyset}] = \int_0^{L_\mathrm{e}} \left(\left(EI + \frac{1}{8} l^2 \mu A\right) \left[\frac{\partial \varphi^{(\emptyset)}}{\partial X}\right]^T \left[\frac{\partial \varphi^{(\emptyset)}}{\partial X}\right] + k_s \mu A [\varphi^{(\emptyset)}]^T [\varphi^{(\emptyset)}] \right) dX \tag{42e}$$

$$[K_w] = \int_0^{L_\mathrm{e}} k_w [\varphi^{(V)}]^T [\varphi^{(V)}] dX \tag{42f}$$

$$[K_p] = \int_0^{L_\mathrm{e}} k_p \left[\frac{\partial \varphi^{(V)}}{\partial X}\right]^T \left[\frac{\partial \varphi^{(V)}}{\partial X}\right] dX \tag{42g}$$

where L_e indicates the length of the finite beam element. The components of the mass matrix $[M]$

$$[M]=\begin{bmatrix}[M^U] & 0 & 0\\ 0 & [M^{\mathrm{V}}] & 0\\ 0 & 0 & [M^{\emptyset}]\end{bmatrix} \tag{43}$$

where $[M_U]$, $[M_V]$ and $[M_{\emptyset}]$ are the contribution of u, v and $\emptyset$ degree of freedom to the mass matrix. Explicit forms of $[M]$ are given as follows

$$[M_U]=\int_0^{L_{\mathrm{e}}}\rho A\left[\varphi^{(U)}\right]^T\left[\varphi^{(U)}\right]dX \tag{44a}$$

$$[M_V]=\int_0^{L_{\mathrm{e}}}\rho A\left[\varphi^{(V)}\right]^T\left[\varphi^{(V)}\right]dX \tag{44b}$$

$$[M_{\emptyset}]=\int_0^{L_{\mathrm{e}}}\rho I\left[\varphi^{(\emptyset)}\right]^T\left[\varphi^{(\emptyset)}\right]dX \tag{44c}$$

The components of the damping matrix $[D]$

$$[D]=\begin{bmatrix}[D^U] & 0 & 0\\ 0 & [D^{\mathrm{V}}] & [D^{V\emptyset}]\\ 0 & [D^{\emptyset V}] & [D^{\emptyset}]\end{bmatrix} \tag{45}$$

where

$$[D^U]=\int_0^{L_{\mathrm{e}}}\eta_1 EA\left[\frac{\partial\varphi^{(U)}}{\partial X}\right]^T\left[\frac{\partial\varphi^{(U)}}{\partial X}\right]dX \tag{46a}$$

$$[D^V]=\int_0^{L_{\mathrm{e}}}\left(\eta_2 k_s\mu A\left[\frac{\partial\varphi^{(V)}}{\partial X}\right]^T\left[\frac{\partial\varphi^{(V)}}{\partial X}\right]+\ \eta_3\frac{1}{8}l^2\mu A\left[\frac{\partial^2\varphi^{(V)}}{\partial X^2}\right]^T\left[\frac{\partial^2\varphi^{(V)}}{\partial X^2}\right]\right)dX \tag{46b}$$

$$\left[D^{V\emptyset}\right]=\left[D^{\emptyset V}\right]^{\mathrm{T}}=\int_0^{L_{\mathrm{e}}}\left(-\eta_2 k_s\mu A\left[\frac{\partial\varphi^{(V)}}{\partial X}\right]^T\left[\varphi^{(\emptyset)}\right]+\ \eta_3\frac{1}{8}l^2\mu A\left[\frac{\partial^2\varphi^{(V)}}{\partial X^2}\right]^T\left[\frac{\partial\varphi^{(\emptyset)}}{\partial X}\right]\right)dX \tag{46c}$$

$$\left[D^{\emptyset}\right]=\int_0^{L_{\mathrm{e}}}\left(\left(\eta_1 EI+\ \eta_3\frac{1}{8}l^2\mu A\right)\left[\frac{\partial\varphi^{(\emptyset)}}{\partial X}\right]^T\left[\frac{\partial\varphi^{(\emptyset)}}{\partial X}\right]+\eta_2 k_s\mu A\left[\varphi^{(\emptyset)}\right]^T\left[\varphi^{(\emptyset)}\right]\right)dX \tag{46d}$$

The load vector $\{F(t)\}$ is expressed as

$$\{F(t)\}=\int_{\mathrm{x}=0}^{L_{\mathrm{e}}}\{\varphi(\mathrm{X})\}^{\mathrm{T}}F(\mathrm{X},t)\ dX \tag{47}$$

The dimensionless quantities can be expressed as

$$\bar{k}_w=\frac{k_w L^4}{EI},\bar{k}_p=\frac{k_p L^2}{EI} \tag{48}$$

where $\bar{k}_w$ is the dimensionless value of Winkler parameter and $\bar{k}_p$ is the dimensionless value of Pasternak parameter.

3. Results and discussions

In this section, various numerical examples are presented and discussed to investigate the forced vibration responses of viscoelastic nanobeam resting on Winkler-Pasternak foundation. In order to determine the effects of the foundation parameters, material length scale parameters and different dimensions of the beam on the forced vibration responses of the nanobeam, result are obtained in conjunctions with the MCST and the CBT. The nanobeam is taken to be made of epoxy (E=1,44 GPa, $\nu = 0.38$, $l = 17.6\,\mu\mathrm{m}$, $\rho = 1600\,\frac{\mathrm{kg}}{\mathrm{m}^3}$, $\mathrm{c} = 14.4\,\mathrm{Pa/s}$). The shear correction factor is taken as $k_s = 0.8922$. In the numerical calculations, the number of finite elements is taken as n=300. In the numerical integrations, five-point Gauss integration rule is used.

Numerical calculations in the time domain are made by using Newmark average acceleration method. The system of linear differential equations which are given by Eq. (40), is reduced to a linear algebraic system of equations by using average acceleration method. The beam is excited by a transverse triangular force impulse (with a peak value 1 μN) modulated by a harmonic function (Fig. 3).

In order to investigate the effect of Winkler parameter on the dynamic responses of the nanobeam in both MCST and CBT, the maximum transverse displacements of the nanobeam obtained for different aspect ratios (L/D) and the material length scale parameter (D/l) in Figs. 4 and 5, respectively for $\bar{k}_p = 0$.

As seen from Fig. 4, the displacement waves which appear after the time interval of the applied impulse force are the excitation wave and primary waves (reflecting waves from the supports of the nanobeam). Also, it is seen Fig. 6 that the displacement waves gradually disappear with increase in time because of the damping effect. It is seen from Fig. 4 that with increase in the aspect ratio, the difference between the results of CBT and MCST decrease considerably. For high values of the aspect ratio, the displacement waves of CBT and MCST interfere with each other. The number of waves for both of CBT and MCST decrease with increase in the aspect ratio. It is observed from Fig. 4 that for the smaller value of aspect ratio, MCST must be used instead of CBT. Also, with increasing the Winkler parameter $\bar{k}_w$, the transverse displacements are decreases, as expected. Increases with the dimensionless stiffness parameter of the Winkler foundation, the differences between results of CBT and MCST decrease significantly. For the smaller ratio of L/D, the MCST must be used instead of the CBT.

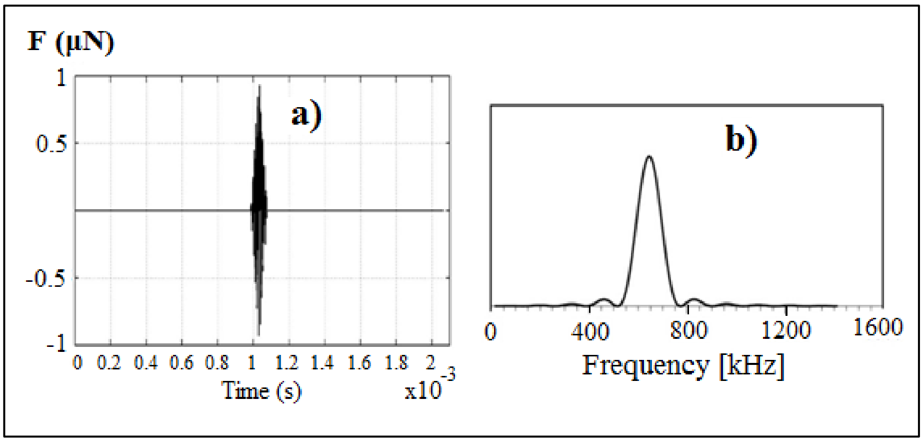

Fig. 3 The shape of the excitation impulse in the (a) time domain and (b) frequency domain

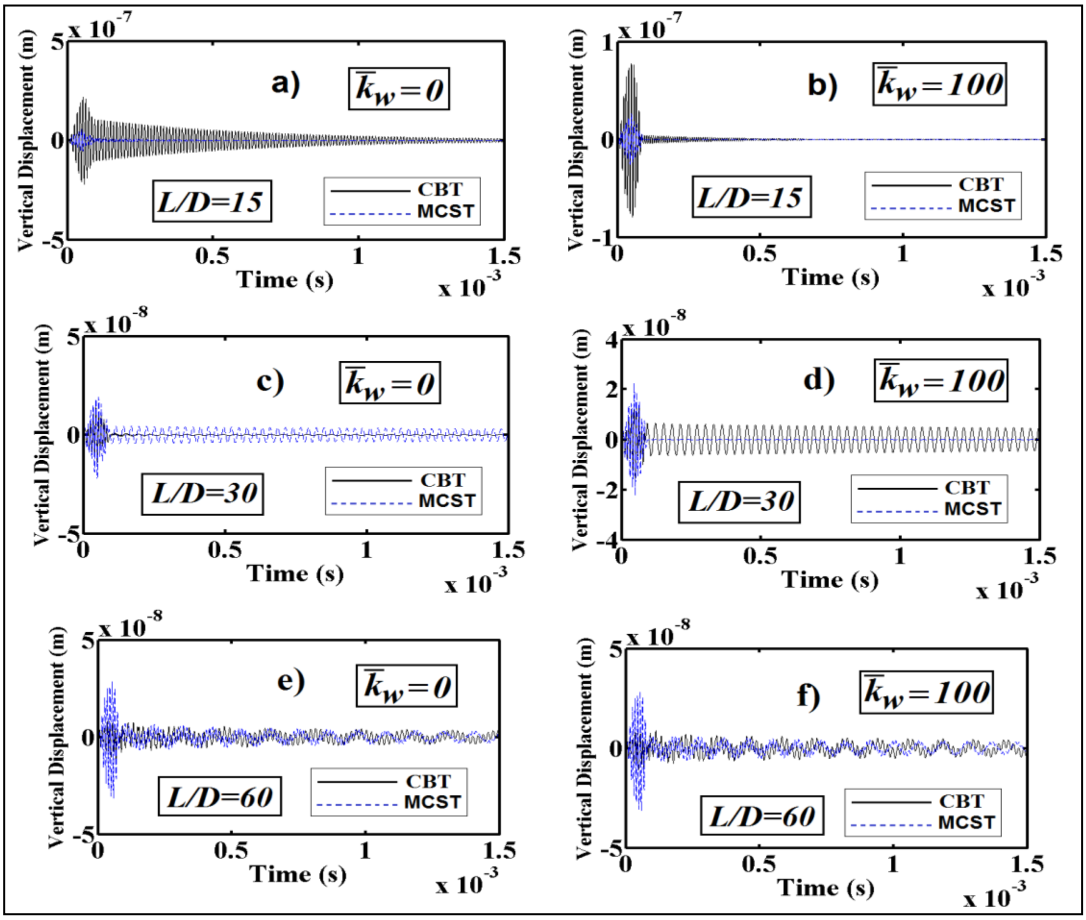

Fig. 4 Effect of Winkler parameter on the transverse displacements for CBT and MCST for different the aspect ratios; (a) $\overline{k}_w$=0 for *L/D*=15, (b) $\overline{k}_w$=100 for *L/D*=15, (c) $\overline{k}_w$=0 for *L/D*=30, (d) $\overline{k}_w$=100 for *L/D*=30, (e) $\overline{k}_w$=0 for *L/D*=60 and (f) $\overline{k}_w$=100 for *L/D*=60

As seen from Fig. 5, one observes that an increase in the material length scale parameter (*D/l*) results in a considerable decrease in the difference between the results of the MCST and CBT for different $\overline{k}_w$. With increase in the *D/l*, the number of displacements waves for both of CBT and MCST decrease significantly. Also, it is seen Fig. 5 that the displacements of the MCST and CBT interfere with each other in the higher value of the $\overline{k}_w$. It is shown results that the Winkler parameter is very effective for mechanical behavior of nanobeam and decreasing in the difference between the MCST and CBT.

Fig. 6 show that the effect of the Pasternak foundation parameter ($\overline{k}_p$) on the maximum transverse displacements for different the material length scale parameter for $\overline{k}_w = 80$.

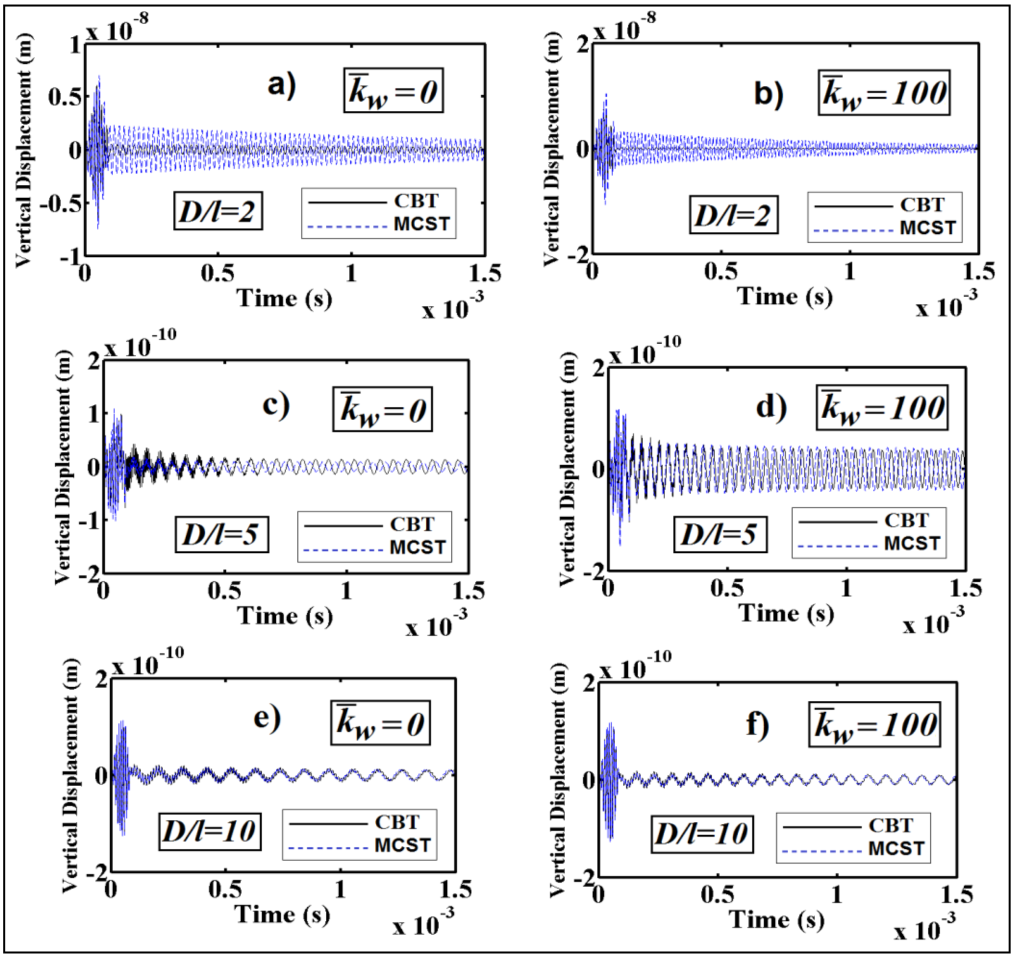

Fig. 5 Effect of Winkler parameter on the transverse displacements for CBT and MCST for different the *D/l* ratios; (a) $\bar{k}_w$=0 for *D/l*=2, (b) $\bar{k}_w$=100 for *D/l*=2, (c) $\bar{k}_w$=0 for *D/l*=5, (d) $\bar{k}_w$=100 for *D/l*=5, (e) $\bar{k}_w$=0 for *D/l*=10 and (f) $\bar{k}_w$=100 for *D/l*=10

As seen from Fig. 6, with increase in the Pasternak parameter, the differences between results of CBT and MCST decrease significantly. Also, the number of displacement waves decrease with increase in the Pasternak parameter. It can be found Figs. 4-6 that the effect of the Winkler parameter more effective than that of the Pasternak parameter. In small values of the foundation parameter, the difference between results of the MCST and CBT is quite noticeable. As seen from figures that increase in the *L/D* and *D/l*, the effect of the foundation parameter on the dynamic responses decrease, considerably.

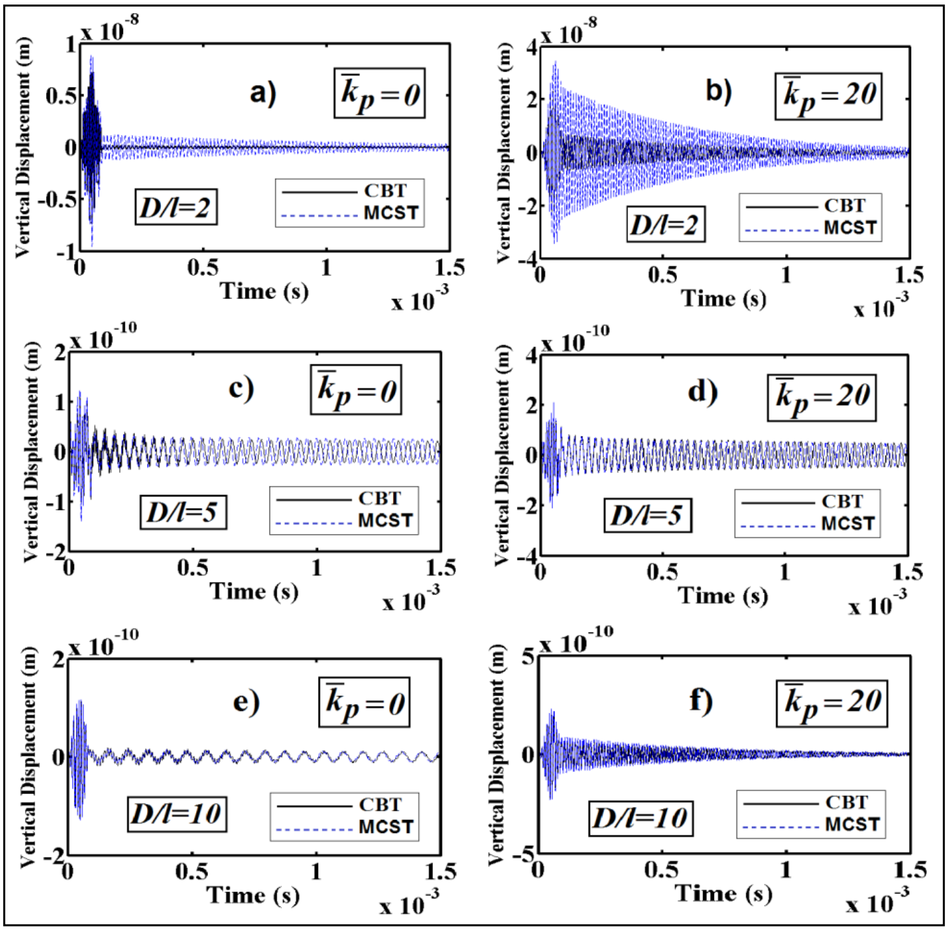

Fig. 6 Effect of Pasternak parameter on the transverse displacements for CBT and MCST for different the *D/l* ratios; (a) $\bar{k}_p$=0 for *D/l*=2, (b) $\bar{k}_p$=20 for *D/l*=2, (c) $\bar{k}_p$=0 for *D/l*=5, (d) $\bar{k}_p$=20 for *D/l*=5, (e) $\bar{k}_p$=0 for *D/l*=10 and (f) $\bar{k}_p$=20 for *D/l*=10

In order to investigate the difference between results of Euler-Bernoulli beam theory (EBT) and Timoshenko beam theory (TBT) the maximum transverse displacements of the nanobeam presented for different aspect ratios (*L/D*) in both CBT and MCST in Figs. 7 and 8, respectively for $\bar{k}_w = 0$, $\bar{k}_p = 0$, *D/l*=5.

It is observed from Figs. 7 and 8 that, , with decrease in the ratio *L/h*, the difference between the results of Euler Bernoulli beam theory and Timoshenko beam theory coincide with each other in both MCST for CBT. the difference between the results of Euler-Bernoulli beam theory and Timoshenko beam theory increases considerably with decreases in the slenderness ratio. Therefore, for small slenderness of beam, Timoshenko beam theory must be used instead of Euler-Bernoulli beam theory because of the effect of the shear stresses on the deformation.

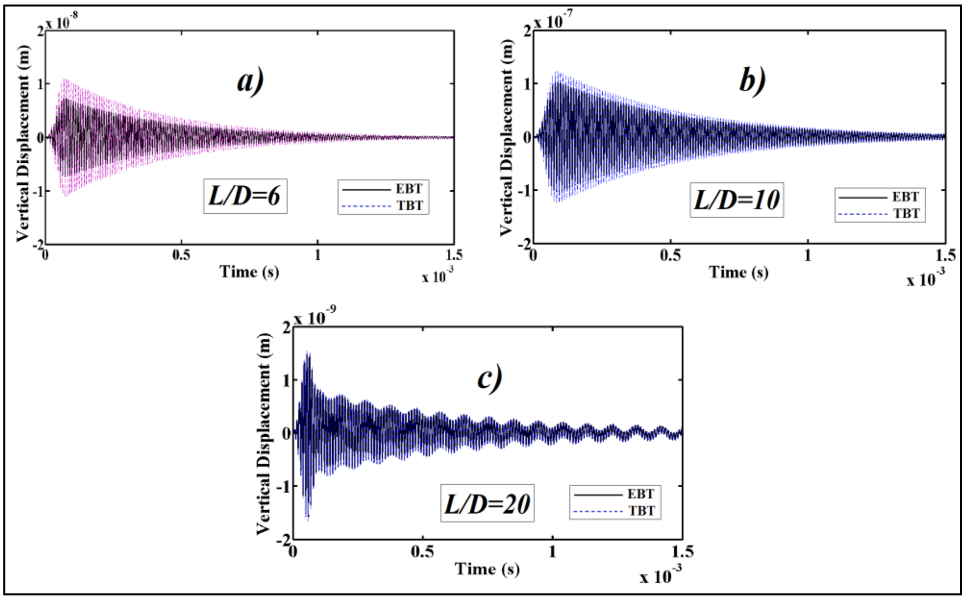

Fig. 7 Effect of aspect ratio on the transverse displacements for CBT for EBT and TBT; (a) *L/D*=6, (b) *L/D*=10 and (c) *L/D*=20

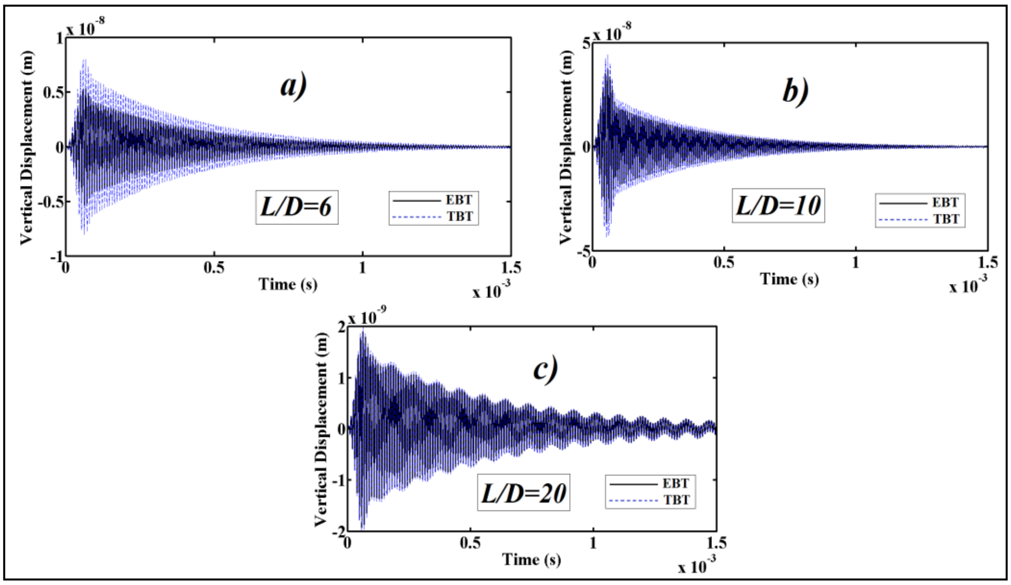

Fig. 8 Effect of aspect ratio on the transverse displacements for MCST for EBT and TBT; (a) *L/D*=6, (b) *L/D*=10 and (c) *L/D*=20

4. Conclusions

Based on the MSCT, the forced vibration behavior of a simple supported viscoelastic nanobeam resting on Winkler-Pasternak elastic foundation is studied with Timoshenko beam theory by using finite element method. The damping effect is considered by using the Kelvin–Voigt viscoelastic model. The obtained system of differential equations is reduced to a linear algebraic equation system and solved in the time domain by using Newmark average acceleration method. Numerical results are presented to investigate the influences the material length scale parameter, aspect ratio, and the foundation parameters on the forced vibration behavior of viscoelastic nanobeams. The difference between the classical beam theory and modified couple stress theory is investigated for forced vibration of viscoelastic nanobeams.

From these results presented and discussed, the main conclusions are as follows:

- The aspect ratio and material parameter have a very important role on the forced vibration responses of the nanobeams.
- The Winkler and Pasternak foundation parameters are very effective for the dynamic response of the nanobeam.
- With increase in the aspect ratio and the dimensionless material length scale parameter, the difference between the dynamic responses of CBT and MCST decrease considerably.
- For high values of the aspect ratio and the dimensionless material length scale parameter, the displacement waves of CBT and MCST interfere with each other.
- The number of displacement waves for both of CBT and MCST decrease with increase in the aspect ratio and the dimensionless material length scale parameter.
- For the smaller ratio of *L/D*, the MCST must be used instead of the CBT.
- Increases with the stiffness parameter of the Winkler and Pasternak foundation, the differences between results of CBT and MCST decrease significantly.
- The Winkler parameter more effective than that of the Pasternak parameter in the dynamic responses of the nanobeams.

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