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Lateral-torsional buckling of prismatic and tapered thin-walled open beams: assessing the influence of pre-buckling deflections

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Abstract. The paper begins by presenting a unified variational approach to the lateral-torsional buckling (LTB) analysis of doubly symmetric prismatic and tapered thin-walled beams with open cross-sections, which accounts for the influence of the pre-buckling deflections. This approach (i) extends the kinematical assumptions usually adopted for prismatic beams, (ii) consistently uses shell membrane theory in general coordinates and (iii) adopts Trefftz's criterion to perform the bifurcation analysis. The proposed formulation is then applied to investigate the influence of the pre-buckling deflections on the LTB behaviour of prismatic and web-tapered I-section simply supported beams and cantilevers. After establishing an interesting analytical result, valid for prismatic members with shear centre loading, several elastic critical moments/loads are presented, discussed and, when possible, also compared with values reported in the literature. These numerical results, which are obtained by means of the Rayleigh-Ritz method, (i) highlight the qualitative differences existing between the LTB behaviours of simply supported beams and cantilevers and (ii) illustrate how the influence of the pre-buckling deflections on LTB is affected by a number of factors, namely (ii_1) the minor-to-major inertia ratio, (ii_2) the beam length, (ii_3) the location of the load point of application and (ii_4) the bending moment diagram shape.

Key words: lateral-torsional buckling; thin-walled beams; prismatic and web-tapered I-beams; pre-buckling deflections; variational formulation; Rayleigh-Ritz method.

1. Introduction

It is a well-known fact that prismatic or tapered beams bent in their stiffer principal plane (i.e., each cross-section is subjected to major axis bending) are prone to *lateral-torsional buckling* (LTB), a bifurcation-type of instability involving a combination of *out-of-plane deflection* and *twisting*. Due to their low minor axis bending and torsional stiffness values, most thin-walled beams with open cross-

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sections are highly susceptible to this buckling phenomenon, which often governs their structural behaviour and load-carrying capacity. When analysing the elastic LTB behaviour of a given beam, it is common practice to neglect the pre-buckling in-plane flexural deflections, i.e., to assume that the beam remains straight up until the onset of buckling. This "classical" approach, which is equivalent to performing a stability analysis of a beam with a linear pre-buckling behaviour, (i) leads to sufficiently accurate results when the ratio between the cross-section major and minor axis inertias is very high and (ii) has been extensively adopted to investigate the LTB behaviour of both prismatic beams (e.g., Timoshenko & Gere 1961, Vlassov 1961 or Trahair 1993) and tapered beams (e.g., Andrade & Camotim 2003, which also includes a fairly substantial literature review on this topic). However, when the cross-section major and minor inertia values are not too far apart, the effect of the (in-plane) pre-buckling deflections on the beam LTB behaviour becomes relevant and neglecting it may lead to a non-negligible underestimation of the corresponding critical load parameter.

The effect of pre-buckling deflections on the LTB behaviour of prismatic beams has been investigated by a number of researchers and it is worth mentioning that most of the earlier studies addressed beams with narrow rectangular cross-sections (e.g., Michell 1899, Pettersson 1952, and Clark & Knoll 1958). Concerning I-section beams, Davidson (1952) derived a transcendental equation, the solution of which provides the elastic critical moment of simply supported doubly symmetric beams under uniform bending. A few years later, Baker *et al.* (1956) published a closed-form solution for this problem, which is based on Davidson's transcendental equation. The investigation concerning the effect of pre-buckling deflections on the stability of equal-flanged prismatic I-beams was extended to a wide range of loading and support conditions by Trahair & Woolcock (1973) and Vacharajittiphan *et al.* (1974), who used the finite integral method to solve the governing differential equilibrium equations. About one decade ago, Pi & Trahair (1992a,b) (i) developed a finite element formulation to analyse the LTB behaviour of singly symmetric I-beams, which takes into account the pre-buckling deflection effects, and (ii) reported numerical applications of this finite element approach. Roberts (1981) derived non-linear strain-displacement expressions for prismatic thin-walled members with arbitrary open cross-sections subjected to bending and torsion, which were subsequently incorporated into a variational formulation intended to analyse the LTB behaviour of beam-columns. Later, this formulation was numerically implemented by means of the Rayleigh-Ritz method and employed to assess the influence of the pre-buckling displacements on the elastic stability of beams (Roberts & Azizian 1983, Roberts & Burt 1985)[1]. Non-linear theories to describe the combined flexural-torsional behaviour of prismatic thin-walled members with arbitrary open cross-sections were developed, independently, by Attard (1986a), Mollmann (1986), Van Erp *et al.* (1988) and Ville de Goyet (1989). All these theories are capable of incorporating the pre-buckling deflections in the LTB analysis of beams. Moreover, Attard (1986b) developed a finite element formulation specifically intended to analyse the beams LTB behaviour, taking into account the influence of the pre-buckling flexural deflections. The beam finite element formulated by Ronagh & Bradford (1999), for the geometrically non-linear analysis of thin-walled prismatic members with open cross-sections, is based on expressions for the first and second variations of the total potential energy identical to the ones derived by Attard (1986a).

The available research work concerning the influence of the pre-buckling deflections on the LTB behaviour of tapered beams is rather scarce and quite recent. Indeed, Ronagh *et al.* (2000a) appear to have been the first to take this influence into account, while deriving an expression for the second

[1]It should be mentioned that Achour & Roberts (2000) presented new strain-displacement relations, which are slightly different from the ones originally proposed by Roberts (1981).

variation of the total potential energy of tapered thin-walled open beams with an arbitrary geometry. However, in spite of the inherent generality of this theory, it was specialised and numerically implemented for doubly symmetric beams only: in a companion paper (Ronagh *et al.* 2000b), a (doubly symmetric) tapered beam finite element was formulated and applied to determine the elastic critical moments of web-tapered simply supported I-beams acted by point loads. It is still worth pointing out that Boissonnade & Muzeau (2001) developed a geometrically non-linear beam finite element, which appears to be capable of including the effect of pre-buckling deflections in the LTB analysis of singly symmetric tapered I-section beams. However, this possibility has not yet been illustrated.

This paper presents a unified variational approach to the LTB analysis of prismatic and tapered thin-walled beams with open cross-sections, which takes into account the influence of the pre-buckling deflections. The proposed formulation extends previous work by the authors on the linear stability behaviour of this type of members (Andrade & Camotim 2003) and its development (i) is based on shell membrane theory in general coordinates, (ii) generalises the kinematical assumptions commonly used for prismatic beams (Vlassov 1961) and (iii) adopts Trefftz's criterion to perform the bifurcation analysis. Moreover, due to space limitations, only doubly symmetric beams are dealt with[2].

The above formulation is then applied to prismatic and web-tapered simply supported I-section beams and cantilevers acted by conservative loads. Aside from validation purposes, this work aims at (i) assessing the influence of the pre-buckling deflections on the LTB behaviour of prismatic and tapered beams and (ii) identifying the role played by the main parameters involved, namely the (ii_1) major-to-minor inertia ratio, (ii_2) beam length, (ii_3) load point of application and (ii_4) bending moment diagram shape. After the demonstration of an analytical result, concerning prismatic beams under shear centre loading, several numerical results are presented and discussed. They consist of elastic critical moments/loads and have been obtained through the Rayleigh-Ritz method. When possible, these results are also compared with values reported in the literature.

2. LTB of doubly symmetric tapered beams: formulation

In this section, one presents a general analytical formulation to analyse the elastic lateral-torsional buckling behaviour of doubly symmetric tapered thin-walled beams with open cross-sections. This formulation both specialises and extends the one recently developed by the authors (Andrade & Camotim 2003), in the sense that it (i) no longer can be applied to singly symmetric beams but (ii) includes the effect of pre-buckling deflections (i.e., it abandons the "classical" linear pre-buckling behaviour assumption).

2.1. Beam and loading description

Fig. 1 shows the undeformed configuration of a typical doubly symmetric thin-walled beam with an open cross-section and length l. A fixed rectangular right-handed Cartesian reference system x, y, z is also shown in Fig. 1, where (i) the x-axis coincides with the undeformed beam centroidal (and shear centre) axis and (ii) the y and z-axes are the cross-section major and minor central axes, respectively. The unit vectors along the Cartesian axes are denoted by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$. Furthermore, let $\mathscr{S}$ and $\mathscr{L}(x)$,

[2]Note that the authors already developed and numerically implemented a more general version of the formulation presented here, which is valid for singly symmetric beams loaded in their symmetry plane (Andrade & Camotim 2002, Andrade 2003).

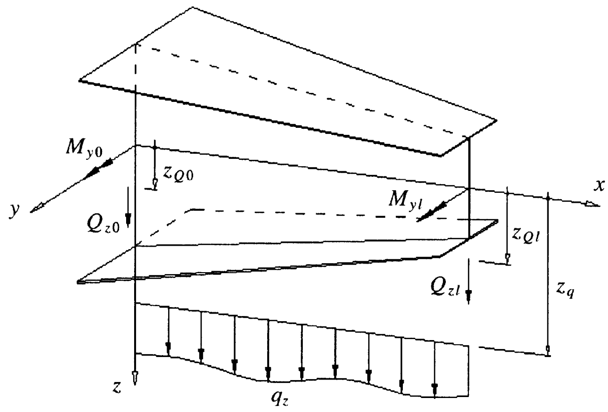

Fig. 1 Tapered thin-walled beam with open cross-section: undeformed configuration, fixed Cartesian axes and external transverse forces and end moments

with $0 \le x \le l$, be the mid-surface and cross-section mid-lines of the undeformed beam.

A material point on the beam mid-surface is identified by its Cartesian coordinates (x, $\bar{y}$, $\bar{z}$) in the undeformed configuration: x specifies the cross-section, whereas $\bar{y}$ and $\bar{z}$ define the location of this point on $\mathscr{L}(x)$. The bar is used to identify a quantity associated with the mid-surface.

Alternatively, the above material point location in the undeformed configuration can be specified by means of curvilinear (Gaussian) coordinates θ^α, defined on $\mathscr{S}$ as follows[3]: $\theta^1 = x$ and θ^2 measures the arc length along the cross-section mid-lines (i.e., along each θ^2-curve), with $\theta^2 = 0$ on the x-axis. According to this alternative description, both $\bar{y}$ and $\bar{z}$ are functions of θ^1 and θ^2. Then, the position vector (relative to the origin of the fixed Cartesian frame) of a generic mid-surface point in the undeformed configuration may be written in the form $\bar{\boldsymbol{r}}(\theta^1, \theta^2) = \theta^1 \boldsymbol{e}_1 + \bar{y}(\theta^1, \theta^2)\boldsymbol{e}_2 + \bar{z}(\theta^1, \theta^2)\boldsymbol{e}_3$.

In this paragraph, one briefly recalls basic concepts and results from the differential geometry of surfaces (e.g., Green & Zerna 1968). The vectors $\boldsymbol{a}_\alpha$, defined on $\mathscr{S}$ by

$$\boldsymbol{a}_1 = \frac{\partial \bar{\boldsymbol{r}}}{\partial \theta^1} = \bar{\boldsymbol{r}}_{,1} = \boldsymbol{e}_1 + \bar{y}_{,1}\boldsymbol{e}_2 + \bar{z}_{,1}\boldsymbol{e}_3 \qquad \boldsymbol{a}_2 = \frac{\partial \bar{\boldsymbol{r}}}{\partial \theta^2} = \bar{\boldsymbol{r}}_{,2} = \bar{y}_{,2}\boldsymbol{e}_2 + \bar{z}_{,2}\boldsymbol{e}_3 \tag{1}$$

are tangent to the coordinate curves and form the *covariant basis of* $\mathscr{S}$ (note that $\boldsymbol{a}_2$ is a unit vector, a direct consequence of the way in which the θ^2 coordinate is defined). The *contravariant basis* (or *dual basis*) $\{\boldsymbol{a}^1, \boldsymbol{a}^2\}$ is defined by the relation

$$\boldsymbol{a}^\alpha \cdot \boldsymbol{a}_\beta = \delta^\alpha_\beta \tag{2}$$

where δ^α_β is the Kronecker symbol. The symmetric second-order tensor with covariant components given by

$$a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta \tag{3}$$

is the *metric tensor of* $\mathscr{S}$, also known as the *first fundamental form of* $\mathscr{S}$. The contravariant components of this surface tensor are

[3]Unless otherwise stated, Greek indices belong to {1, 2} and Einstein's summation convention is adopted.

$$a^{\alpha\beta} = \boldsymbol{a}^{\alpha} \cdot \boldsymbol{a}^{\beta} = (a_{\alpha\beta})^{-1} \tag{4}$$

Expressions (3) and (4) yield

$$a_{11} = 1 + \bar{y}_{,1}^{2} + \bar{z}_{,1}^{2} \qquad a_{12} = a_{21} = \bar{y}_{,1}\bar{y}_{,2} + \bar{z}_{,1}\bar{z}_{,2} \qquad a_{22} = \bar{y}_{,2}^{2} + \bar{z}_{,2}^{2} = 1 \tag{5}$$

$$a^{11} = \frac{a_{22}}{a} \qquad a^{12} = a^{21} = -\frac{a_{12}}{a} \qquad a^{22} = \frac{a_{11}}{a} \tag{6}$$

where $a = \det(a_{\alpha\beta}) = 1+(\bar{y}_{,1}\bar{z}_{,2} - \bar{y}_{,2}\bar{z}_{,1})^2$. The *area element dA on $\mathscr{S}$* reads

$$dA = \|\boldsymbol{a}_1 \times \boldsymbol{a}_2\| d\theta^1 d\theta^2 = \sqrt{a}\, d\theta^1 d\theta^2 \tag{7}$$

Finally, one notes that the vector

$$\boldsymbol{a}_3 = \frac{\boldsymbol{a}_2 \times \boldsymbol{a}_1}{\|\boldsymbol{a}_2 \times \boldsymbol{a}_1\|} \tag{8}$$

has Euclidean norm $\|\boldsymbol{a}_3\| = 1$ and is normal to $\mathscr{S}$. A word of caution is in order at this point: in tapered beams, the base vectors $\boldsymbol{a}_\alpha$ (or $\boldsymbol{a}^\alpha$) are not orthogonal (in other words, the Gaussian coordinate system θ^α is not orthogonal, an issue apparently overlooked by Ronagh *et al.* 2002a). It is therefore convenient to introduce orthonormal vectors $\boldsymbol{A}_I$ and $\boldsymbol{A}_{II}$, spanning the tangent plane to $\mathscr{S}$ and exhibiting the following properties: (i) $\boldsymbol{A}_{II} = \boldsymbol{a}_2$ and (ii) $\boldsymbol{A}_{II} \times \boldsymbol{A}_I = \boldsymbol{a}_3$ (Wilde 1968). From these conditions, one obtains

$$\boldsymbol{A}_I = \frac{1}{\sqrt{a}}\boldsymbol{a}_1 - \frac{a_{12}}{\sqrt{a}}\boldsymbol{a}_2 \tag{9}$$

Concerning the end support and loading conditions, only the following two cases are addressed in this work: (i) simply supported beams subjected to transverse loads and/or end moments and (ii) cantilevers acted by transverse loads. The end sections of a simply supported beam are prevented from deflecting along y and z, but are free to rotate about both these axes. In addition, they are restrained against torsion and may warp freely. The cantilevers are assumed to be fully built-in at the support (i.e., all displacements are prevented at this end section) and completely free at the other end. The external loads, generically shown in Fig. 1, are deemed conservative and proportional to a single parameter λ. The conservative character of these loads is ensured by the fact that (i) the transverse forces follow the beam deformation, always retaining their original direction, and (ii) the end moments M_{y0} and M_{yl} follow the corresponding end section rotation about the x-axis, thus remaining normal to this axis.

2.2. Kinematics

The one-dimensional theory characterising LTB is derived by regarding thin-walled open beams as kinematically constrained membrane shells. These kinematical constraints extend the classical Vlassov's assumptions (Vlassov 1961), adopted in the context of prismatic beams, and may be stated as follows (Andrade & Camotim 2003):

(i) The projection of a cross-section mid-line on a plane normal to the (fixed) x-axis retains its shape and dimensions throughout the whole deformation process.

(ii) The mid-surface shear strains (involving fibres originally along the orthonormal vectors $\boldsymbol{A}_I$ and $\boldsymbol{A}_{II}$) are negligible.

It is important to stress that the first constraint precludes the occurrence of any local or distortional instability phenomena. Furthermore, it is assumed throughout the derivations that the strains and the displacements along the x-axis are small (i.e., the strain components and the derivatives of this displacement component are negligible when compared with 1).

According to the above kinematical constraints, the deformed configuration of any given cross-section mid-line can be regarded as the result of (i) an in-plane rigid-body motion, followed by (ii) displacements in the x-axis direction, due to bending and warping. The rigid-body motion can still be decomposed into the successive application of (i) a rotation Φ about the (fixed) x-axis and (ii) a translation with components along y (V) and z (W). Therefore, the transverse displacement field $\overline{V}\boldsymbol{e}_2 + \overline{W}\boldsymbol{e}_3$ of the beam mid-surface is defined by

$$\overline{V}(\theta^1, \theta^2) = V(\theta^1) - \bar{y}(\theta^1, \theta^2)(1 - \cos\Phi(\theta^1)) - \bar{z}(\theta^1, \theta^2)\sin\Phi(\theta^1) \tag{10}$$

$$\overline{W}(\theta^1, \theta^2) = W(\theta^1) + \bar{y}(\theta^1, \theta^2)\sin\Phi(\theta^1) - \bar{z}(\theta^1, \theta^2)(1 - \cos\Phi(\theta^1)) \tag{11}$$

The covariant components of the *Green-St. Venant membrane strain tensor* in the curvilinear coordinate system θ^α (i.e., half of the *change of metric tensor*, associated with the mid-surface displacement field) are given by (e.g., Green & Zerna 1968).

$$\bar{e}_{\alpha\beta}(\theta^1, \theta^2) = \frac{1}{2}(\boldsymbol{a}_\alpha \cdot \overline{\boldsymbol{U}}_{,\beta} + \boldsymbol{a}_\beta \cdot \overline{\boldsymbol{U}}_{,\alpha} + \overline{\boldsymbol{U}}_{,\alpha} \cdot \overline{\boldsymbol{U}}_{,\beta}) \tag{12}$$

where $\overline{\boldsymbol{U}} = \overline{U}\boldsymbol{e}_1 + \overline{V}\boldsymbol{e}_2 + \overline{W}\boldsymbol{e}_3$ is the mid-surface displacement field. The shear strain involving fibres originally along the orthonormal vectors $\boldsymbol{A}_I$ and $\boldsymbol{A}_{II}$ reads

$$\bar{e}_{I\,II} = \bar{e}_{II\,I} = \frac{1}{\sqrt{a}}(\bar{e}_{12} - a_{12}\bar{e}_{22}) \tag{13}$$

After eliminating negligible terms, this last equation yields

$$\bar{e}_{I\,II} = \bar{e}_{II\,I} = \frac{1}{2\sqrt{a}}[\overline{U}_{,2} + \bar{y}_{,2}A + \bar{z}_{,2}B + (\bar{y}\bar{z}_{,2} - \bar{z}\bar{y}_{,2})\Phi_{,1}] \tag{14}$$

where $A = V_{,1}\cos\Phi + W_{,1}\sin\Phi$ and $B = W_{,1}\cos\Phi - V_{,1}\sin\Phi$. Then, the second kinematical constraint ($\bar{e}_{I\,II} = \bar{e}_{II\,I} = 0$) provides

$$\overline{U}_{,2} = -\bar{y}_{,2}A - \bar{z}_{,2}B - (\bar{y}\bar{z}_{,2} - \bar{z}\bar{y}_{,2})\Phi_{,1} \tag{15}$$

and, upon integration with respect to θ^2, leads to

$$\overline{U} = U - \bar{y}A - \bar{z}B - \varpi\Phi_{,1} \tag{16}$$

where

$$U(\theta^1) = \overline{U}(\theta^1, 0) + \bar{y}(\theta^1, 0)A + \bar{z}(\theta^1, 0)B \tag{17}$$

$$\varpi(\theta^1, \theta^2) = \int_{\mathscr{C}(\theta^1, \theta^2)} [\bar{y}(\theta^1, s)\bar{z}_{,s}(\theta^1, s) - \bar{z}(\theta^1, s)\bar{y}_{,s}(\theta^1, s)]\, ds \tag{18}$$

and $\mathscr{C}(\theta^1, \theta^2)$ is the segment of $\mathscr{L}(\theta^1)$ comprised between the centroid ($\theta^2 = 0$) and the point under consideration, defined by the Gaussian coordinates θ^1, θ^2. Note that the restriction of $\overline{\omega}$ to $\mathscr{L}(\theta^1)$ represents the sectorial coordinate having both its origin and pole at the centroid.

Incorporating the previous results in Eq. (12) and neglecting $\overline{U}_{,1}$ (in face of 1) yields

$$\bar{e}_{11} = U_{,1} - \bar{y}(A_{,1} - B\Phi_{,1}) - \bar{z}(B_{,1} + A\Phi_{,1}) - \varpi\Phi_{,11} + \frac{1}{2}[V_{,1}^2 + W_{,1}^2 + (\bar{y}^2 + \bar{z}^2)\Phi_{,1}^2] - \psi\Phi_{,1} \quad (19)$$

$$\bar{e}_{12} = \bar{e}_{21} = 0 \qquad \bar{e}_{22} = \frac{1}{2}[\bar{y}_{,2}(A - \bar{z}\Phi_{,1}) + \bar{z}_{,2}(B + \bar{y}\Phi_{,1})]^2 \quad (20)$$

where the function

$$\psi(\theta^1, \theta^2) = \varpi_{,1} + \bar{y}_{,1}\bar{z} - \bar{z}_{,1}\bar{y} \quad (21)$$

appearing in Eq. (19), stems from the cross-section variation and is responsible for the *qualitative differences* exhibited by the LTB behaviours of tapered and prismatic beams. This last statement means that, in tapered beams with $\psi \neq 0$, piecewise prismatic models will not converge to the correct LTB solution (Andrade & Camotim 2003). It is worth noting that Eq. (19) agrees with the one derived by Ronagh *et al.* (2000a), but differs substantially from the one obtained by Rajasekaran (1994), on the basis of inconsistent approximations - the author neglects some non-linear terms having the same order of magnitude as others that are retained.

2.3. Total potential energy of the beam - load system

One assumes that the material constituting the beam is a *St. Venant - Kirchhoff material* (which, by definition, is *homogeneous*, *isotropic* and *hyperelastic*; the fact that the undeformed configuration is a *natural state* is also implied - e.g., Ciarlet 1988), with Young modulus E and Poisson ratio ν.

The membrane strain energy is given by

$$\frac{1}{2}\int_{\mathscr{S}} \bar{n}^{\alpha\beta}\bar{e}_{\alpha\beta}dA = \frac{1}{2}\int_{\mathscr{S}} (\bar{n}^{11}\bar{e}_{11} + \bar{n}^{22}\bar{e}_{22})dA \quad (22)$$

where the $\bar{n}^{\alpha\beta}$ are the contravariant components of the *tensor of membrane forces*, work-conjugate to the Green-St. Venant strains $\bar{e}_{\alpha\beta}$, and dA is given by Eq. (7). One has, under plane stress conditions (Green & Zerna 1968).

$$\bar{n}^{11} = Et\{(a^{11})^2\bar{e}_{11} + [(1-\nu)(a^{12})^2 + \nu a^{11}a^{22}]\bar{e}_{22}\} \quad (23)$$

$$\bar{n}^{22} = Et\{[(1-\nu)(a^{12})^2 + \nu a^{11}a^{22}]\bar{e}_{11} + (a^{22})^2\bar{e}_{22}\} \quad (24)$$

where t is the wall thickness and the approximation $1 - \nu^2 \approx 1$ was adopted[4].

The strain energy associated with uniform (or St. Venant) torsion, which is disregarded in the membrane shell model, is added separately and evaluated through

$$\frac{G}{2}\int_0^l J\Phi_{,1}^2 d\theta^1 \quad (25)$$

[4]If one does not wish to adopt this approximation, often used in the context of beam theories, it suffices to replace E by $E/(1-\nu^2)$ in the relevant equations.

where $G = E/[2(1+\nu)]$ is the shear modulus (i.e., one uses the expression valid for prismatic beams, though taking into account the variation of the cross-sectional property J along the beam length).

Finally, the potential energy of the external loads (see Fig. 1) is defined by

$$\mathscr{V}_e = -\int_0^l q_z[W - z_q(1-\cos\Phi)]d\theta^1 - Q_{z0}[W(0) - z_{Q0}(1-\cos\Phi(0))] -$$
$$- Q_{zl}[W(l) - z_{Ql}(1-\cos\Phi(l))] + M_{y0}(W_{,1}(0)\cos\Phi(0) - V_{,1}(0)\sin\Phi(0)) +$$
$$+ M_{yl}(W_{,1}(l)\cos\Phi(l) - V_{,1}(l)\sin\Phi(l)) \tag{26}$$

2.4. Fundamental equilibrium path

In a fundamental equilibrium state, associated with a given load parameter value λ, the beam is subjected solely to major axis bending and its deformed configuration is thus characterized by $U = U^f$, $V = V^f = 0$, $W = W^f$ and $\Phi = \Phi^f = 0$.

In order to account for the effect of pre-buckling deflections on LTB, one has to put aside the hypothesis of a linear pre-buckling behaviour. Nevertheless, it is assumed that there exists a linear relationship between (i) the membrane forces and stress resultants in a fundamental equilibrium state and (ii) the displacement derivatives. Therefore, one has

$$\bar{n}^{11f}(\theta^1, \theta^2, \lambda) = Et(a^{11})^2(U^f_{,1} - \bar{z}W^f_{,11}) \tag{27}$$

$$\bar{n}^{22f}(\theta^1, \theta^2, \lambda) = Et\left[(a^{12})^2 + \frac{\nu}{a}\right](U^f_{,1} - \bar{z}W^f_{,11}) \tag{28}$$

and the condition of null axial force on the fundamental path yields (note that $\boldsymbol{a}_2 \cdot \boldsymbol{e}_1 = 0$)

$$N^f(\theta^1, \lambda) = \int_{\mathscr{L}(\theta^1)} (a^{11})^{-1/2}(\bar{n}^{11f}\boldsymbol{a}_1 + \bar{n}^{12f}\boldsymbol{a}_2)\cdot \boldsymbol{e}_1 d\theta^2 = EA^* U^f_{,1} = 0 \quad \Rightarrow U^f_{,1} = 0 \tag{29}$$

where $A^*(\theta^1) = \int_{\mathscr{L}(\theta^1)} t^* d\theta^2$ and $t^* = t[1 + (\bar{y}_{,1}\bar{z}_{,2} - \bar{y}_{,2}\bar{z}_{,1})^2]^{-3/2}$ ($\leq t$). Moreover, the bending moment distribution in a fundamental state is defined by

$$M^f_y(\theta^1, \lambda) = \int_{\mathscr{L}(\theta^1)} \bar{z}(a^{11})^{-1/2}(\bar{n}^{11f}\boldsymbol{a}_1 + \bar{n}^{12f}\boldsymbol{a}_2)\cdot \boldsymbol{e}_1 d\theta^2 = -EI^*_y W^f_{,11} \tag{30}$$

with $I^*_y(\theta^1) = \int_{\mathscr{L}(\theta^1)} \bar{z}^2 t^* \, d\theta^2$.

2.5. Bifurcation analysis

Let $u(\theta^1)$, $v(\theta^1)$, $w(\theta^1)$ and $\phi(\theta^1)$ denote kinematically admissible variations of the (generalized) displacements from a fundamental equilibrium state, which are independent from λ. Then, one defines an adjacent configuration by

$$U(\theta^1,\lambda) = U^f(\theta^1,\lambda)+u(\theta^1) = u(\theta^1) \qquad V(\theta^1,\lambda) = V^f(\theta^1,\lambda)+v(\theta^1) = v(\theta^1)$$

$$W(\theta^1,\lambda) = W^f(\theta^1,\lambda)+w(\theta^1) \qquad \Phi(\theta^1,\lambda) = \Phi^f(\theta^1,\lambda)+\phi(\theta^1) = \phi(\theta^1) \tag{31}$$

and, according to Trefftz's criterion (e.g., Bazant & Cedolin 1991 or Reis & Camotim 2001), the bifurcation points on the fundamental path are identified by the stationarity condition

$$\delta(\delta^2\Pi) = 0 \tag{32}$$

with respect to all u, v, w and ϕ, where $\delta^2\Pi$ is the second-order term of the Taylor series expansion, about a fundamental state, of the beam total potential energy (i.e., the second variation of Π). Functional $\delta^2\Pi$ may be expressed in the form (Andrade & Camotim 2003)

$$\delta^2\Pi = \int_{\mathscr{S}}\{\bar{n}^{11f}\delta^2\bar{e}_{11}+\bar{n}^{22f}\delta^2\bar{e}_{22}+Et/2\{(a^{11})^2(\delta\bar{e}_{11})^2+(a^{22})^2(\delta\bar{e}_{22})^2+$$
$$+2[(1-\nu)(a^{12})^2+\nu a^{11}a^{22}]\delta\bar{e}_{11}\delta\bar{e}_{22}\}\}dA+G/2\int_0^l J\phi_{,1}^2 d\theta^1+\delta^2\mathscr{V}_e \tag{33}$$

where the variations of the membrane strains and external load potential energy are given by

$$\delta\bar{e}_{11} = u_{,1}-\bar{y}(v_{,11}+W^f_{,11}\phi)-\bar{z}w_{,11}-\bar{\omega}\phi_{,11}-\psi\phi_{,1}+W^f_{,1}w_{,1} \tag{34}$$

$$\delta\bar{e}_{22} = \bar{y}_{,2}\bar{z}_{,2}W^f_{,1}(v_{,1}+W^f_{,1}\phi-\bar{z}\phi_{,1})+\bar{z}^2_{,2}W^f_{,1}(w_{,1}+\bar{y}\phi_{,1}) \tag{35}$$

$$\delta^2\bar{e}_{11} = -\bar{y}w_{,11}\phi+\bar{z}\left(v_{,11}\phi+\frac{1}{2}W^f_{,11}\phi^2\right)+\frac{1}{2}[v_{,1}^2+w_{,1}^2+(\bar{y}^2+\bar{z}^2)\phi_{,1}^2] \tag{36}$$

$$\delta^2\bar{e}_{22} = \frac{1}{2}[\bar{\omega}_{,2}\phi_{,1}+\bar{y}_{,2}(v_{,1}+W^f_{,1}\phi)+\bar{z}_{,2}w_{,1}]^2+\bar{z}_{,2}W^f_{,1}\phi\left[\bar{y}_{,2}w_{,1}-\frac{1}{2}\bar{z}_{,2}(W^f_{,1}\phi+2v_{,1})\right] \tag{37}$$

$$\delta^2\mathscr{V}_e = \frac{z_q}{2}\int_0^l q_z\phi^2 d\theta^1+\frac{z_{Q0}}{2}Q_{z0}\phi(0)^2+\frac{z_{Ql}}{2}Q_{zl}\phi(l)^2-$$
$$-\frac{M_{y0}}{2}(2v_{,1}(0)\phi(0)+W^f_{,1}(0,\lambda)\phi(0)^2)-\frac{M_{yl}}{2}(2v_{,1}(l)\phi(l)+W^f_{,1}(l,\lambda)\phi(l)^2) \tag{38}$$

If one assumes that the contribution of the membrane forces $\bar{n}^{22}$ is negligible (one of the basic assumptions of elementary beam theory), then $\delta^2\Pi$ becomes

$$\delta^2\Pi = \frac{E}{2}\int_0^l[A^*(u_{,1}+W^f_{,1}w_{,1})^2+I^*_y w_{,11}^2+I^*_z(v_{,11}+W^f_{,11}\phi)^2+I^*_\omega\phi_{,11}^2+I^*_\psi\phi_{,1}^2+$$
$$+2I^*_{\omega\psi}\phi_{,1}\phi_{,11}]d\theta^1+\frac{G}{2}\int_0^l J\phi_{,1}^2 d\theta^1+\frac{1}{2}\int_0^l M^f_y(2v_{,11}\phi+W^f_{,11}\phi^2)d\theta^1+$$
$$+\frac{z_q}{2}\int_0^l q_z\phi^2 d\theta^1+\frac{z_{Q0}}{2}Q_{z0}\phi(0)^2+\frac{z_{Ql}}{2}Q_{zl}\phi(l)^2-$$

$$-\frac{M_{y0}}{2}(2v_{,1}(0)\phi(0) + W^f_{,1}(0,\lambda)\phi(0)^2) - \frac{M_{yl}}{2}(2v_{,1}(l)\phi(l) + W^f_{,1}(l,\lambda)\phi(l)^2) \tag{39}$$

where the functions

$$A^*(\theta^1) = \int_{\mathscr{L}(\theta^1)} t^* d\theta^2 \qquad I^*_y(\theta^1) = \int_{\mathscr{L}(\theta^1)} \bar{z}^2 t^* d\theta^2 \qquad I^*_z(\theta^1) = \int_{\mathscr{L}(\theta^1)} \bar{y}^2 t^* d\theta^2$$

$$I_\omega{}^*(\theta^1) = \int_{\mathscr{L}(\theta^1)} \bar{\omega}^2 t^* d\theta^2 \qquad I^*_\psi(\theta^1) = \int_{\mathscr{L}(\theta^1)} \psi^2 t^* d\theta^2 \qquad I^*_{\omega\psi}(\theta^1) = \int_{\mathscr{L}(\theta^1)} \bar{\omega}\psi t^* d\theta^2 \tag{40}$$

are geometrical properties of the beam. Notice also that the orthogonality conditions

$$\int_{\mathscr{L}(\theta^1)} \bar{y} t^* d\theta^2 = 0 \qquad \int_{\mathscr{L}(\theta^1)} \bar{z} t^* d\theta^2 = 0 \qquad \int_{\mathscr{L}(\theta^1)} \varpi t^* d\theta^2 = 0$$

$$\int_{\mathscr{L}(\theta^1)} \psi t^* d\theta^2 = 0 \qquad \int_{\mathscr{L}(\theta^1)} \bar{y}\bar{z} t^* d\theta^2 = 0 \qquad \int_{\mathscr{L}(\theta^1)} \varpi\bar{y} t^* d\theta^2 = 0$$

$$\int_{\mathscr{L}(\theta^1)} \varpi\bar{z} t^* d\theta^2 = 0 \qquad \int_{\mathscr{L}(\theta^1)} \psi\bar{y} t^* d\theta^2 = 0 \qquad \int_{\mathscr{L}(\theta^1)} \psi\bar{z} t^* d\theta^2 = 0 \tag{41}$$

which hold for doubly symmetric beams, were used in derivations.

From a mathematical viewpoint, Eq. (32) is the variational (weak) statement of a non-linear eigenvalue problem. By applying standard Calculus of Variations techniques (e.g., Courant & Hilbert 1953), one is led to the strong form of this problem, defined by the differential equations in $]0,l[$ (Euler-Lagrange equations of $\delta^2\Pi$)

$$E[A^*(u_{,1} + W^f_{,1}w_{,1})]_{,1} = 0 \tag{42}$$

$$E[I^*_z(v_{,11} + W^f_{,11}\phi)]_{,11} + (M^f_y\phi)_{,11} = 0 \tag{43}$$

$$E[A^* W^f_{,1}(u_{,1} + W^f_{,1}w_{,1})]_{,1} - E(I^*_y w_{,11})_{,11} = 0 \tag{44}$$

$$EI^*_z W^f_{,11}(v_{,11} + W^f_{,11}\phi) + M^f_y(v_{,11} + W^f_{,11}\phi) + z_q q_z\phi - E(I^*_\psi\phi_{,1} + I^*_{\omega\psi}\phi_{,11})_{,1} -$$
$$-G(J\phi_{,1})_{,1} + E(I^*_\omega\phi_{,11} + I^*_{\omega\psi}\phi_{,1})_{,11} = 0 \tag{45}$$

and the corresponding (essential and natural) boundary conditions. For simply supported beams and cantilevers, the latter take the form:

(i) Simply supported beams[5] acted by transverse loads and/or end moments

$$u(0) = 0 \qquad E\,A^*(l)(u_{,1}(l) + W^f_{,1}(l,\lambda)w_{,1}(l)) = 0 \tag{46}$$

$$v(0) = v(l) = 0 \tag{47}$$

$$EI^*_z(0)(v_{,11}(0) + W^f_{,11}(0,\lambda)\phi(0)) = EI^*_z(l)(v_{,11}(l) + W^f_{,11}(l,\lambda)\phi(l)) = 0 \tag{48}$$

[5]Without loss of generality, one assumes that the longitudinal displacement is prevented at the end section defined by $\theta^1 = 0$.

$$w(0) = w(l) = 0 \qquad EI_y^*(0)w_{,11}(0) = EI_y^*(l)w_{,11}(l) = 0 \tag{49}$$

$$\phi(0) = \phi(l) = 0 \tag{50}$$

$$E(I_\omega^*(0)\phi_{,11}(0) + I_{\omega\psi}^*(0)\phi_{,1}(0)) = E(I_\omega^*(l)\phi_{,11}(l) + I_{\omega\psi}^*(l)\phi_{,1}(l)) = 0 \tag{51}$$

(ii) Cantilevers[6] acted by transverse loads

$$u(0) = 0 \qquad EA^*(l)(u_{,1}(l) + W_{,1}^f(l,\lambda)w_{,1}(l)) = 0 \tag{52}$$

$$v(0) = 0 \qquad \{E[I_z^*(v_{,11} + W_{,11}^f\phi)]_{,1} + (M_y^f\phi)_{,1}\}_{\theta^1 = l} = 0 \tag{53}$$

$$v_{,1}(0) = 0 \qquad EI_z^*(l)(v_{,11}(l) + W_{,11}^f(l,\lambda)\phi(l)) = 0 \tag{54}$$

$$w(0) = 0 \qquad E\,A^*(l)W_{,1}^f(l,\lambda)(u_{,1}(l) + W_{,1}^f(l,\lambda)w_{,1}(l)) - E[(I_y^*w_{,11})_{,1}]_{\theta^1 = l} = 0 \tag{55}$$

$$w_{,1}(0) = 0 \qquad EI_y^*(l)w_{,11}(l) = 0 \tag{56}$$

$$\phi(0) = 0 \qquad \phi_{,1}(0) = 0 \tag{57}$$

$$EI_\psi^*(l)\phi_{,1}(l) + EI_{\omega\psi}^*(l)\phi_{,11}(l) + GJ(l)\phi_{,1}(l) - E[(I_\omega^*\phi_{,11})_{,1} + (I_{\omega\psi}^*\phi_{,1})_{,1}]_{\theta^1 = l} + \\ + z_{Ql}Q_{zl}\phi(l) = 0 \tag{58}$$

$$E(I_\omega^*(l)\phi_{,11}(l) + I_{\omega\psi}^*(l)\phi_{,1}(l)) = 0 \tag{59}$$

The non-linear character of the above eigenvalue problem stems from the presence of terms containing derivatives of the pre-buckling deflections W^f. Indeed, neglecting these terms, which means disregarding the influence of the pre-buckling deflections on the bifurcation load parameters and buckling modes, amounts to linearising the eigenvalue problem and leads to the "classical" (linear) beam stability analysis, based on the linear pre-buckling behaviour assumption (Andrade & Camotim 2003).

Before closing this sub-section, one shows that the eigenvalue problem just defined can be recast in terms of (i) the load parameter λ and (ii) a single unknown function - the torsional rotation ϕ. Indeed, one readily sees that the integration of Eqs. (42) and (44), together with the boundary conditions (46) and (49) (simply supported beams) or (52), (55) and (56) (cantilevers), yields $u = w = 0$, a result that constitutes the beam counterpart of the inextensional flexural buckling of columns. Then, by considering (i) Eq. (43), together with the boundary conditions (48) and (50) (simply supported beams) or (53_2) and (54_2) (cantilevers), and (ii) Eq. (30)[7], it is possible to conclude that $v_{,11}$ and ϕ are not independent, since they are related by the equation

$$v_{,11} = -\frac{M_y^f}{EI_z^*}\left(1 - \frac{I_z^*}{I_y^*}\right)\phi \tag{60}$$

[6]Again without sacrificing generality, we assume that the built-in section corresponds to $\theta^1 = 0$.
[7]In cantilevers acted by transverse loads, one also uses the fact that $M_y^f(l,\lambda) = 0,\ \forall\lambda$.

Consequently, one reaches the desired result: functional (39) can be rewritten as

$$\delta^2 \Pi = \frac{1}{2}\int_0^l \left[EI_\omega^* \phi_{,11}^2 + EI_\psi^* \phi_{,1}^2 + 2EI_{\omega\psi}^* \phi_{,1}\phi_{,11} + GJ\phi_{,1}^2 - \frac{M_y^{f2}}{EI_z^*}\left(1 - \frac{I_z^*}{I_y^*}\right)\phi^2 + \right.$$

$$\left. + z_q q_z \phi^2 \right] d\theta^1 + \frac{z_{Ql}}{2} Q_{zl} \phi(l)^2 \tag{61}$$

and Eqs. (42)-(59) are replaced by the differential equation

$$-\frac{M_y^{f2}}{EI_z^*}\left(1 - \frac{I_z^*}{I_y^*}\right)\phi + z_q q_z \phi - E(I_\psi^* \phi_{,1} + I_{\omega\psi}^* \phi_{,11})_{,1} - G(J\phi_{,1})_{,1} + E(I_\omega^* \phi_{,11} + I_{\omega\psi}^* \phi_{,1})_{,11} = 0 \tag{62}$$

subjected to the following boundary conditions:

(i) Simply supported beams acted by transverse loads and/or end moments

$$\phi(0) = \phi(l) = 0 \tag{63}$$

$$E(I_\omega^*(0)\phi_{,11}(0) + I_{\omega\psi}^*(0)\phi_{,1}(0)) = E(I_\omega^*(l)\phi_{,11}(l) + I_{\omega\psi}^*(l)\phi_{,1}(l)) = 0 \tag{64}$$

(ii) Cantilevers acted by transverse loads

$$\phi(0) = 0 \qquad \phi_{,1}(0) = 0 \tag{65}$$

$$EI_\psi^*(l)\phi_{,1}(l) + EI_{\omega\psi}^*(l)\phi_{,11}(l) + GJ(l)\phi_{,1}(l) - E[(I_\omega^*\phi_{,11})_{,1} + (I_{\omega\psi}^*\phi_{,1})_{,1}]_{\theta^1 = l} +$$

$$+ z_{Ql} Q_{zl} \phi(l) = 0 \tag{66}$$

$$E(I_\omega^*(l)\phi_{,11}(l) + I_{\omega\psi}^*(l)\phi_{,1}(l)) = 0 \tag{67}$$

2.6. Prismatic beams

In the particular case of prismatic beams, the general formulation just presented can be considerably simplified. In fact, functions $\bar{y}$, $\bar{z}$ and ω cease to depend on θ^1, which implies that (i) ψ is identically null and (ii) the fictitious thickness t^* coincides with the actual thickness t, which is also a function of θ^2 alone. Then, (i) $A^* = A$ is the cross-section area, (ii) $I_y^* = I_y$ and $I_z^* = I_z$ are the major and minor moments of inertia, (iii) $I_\omega^* = I_w$ is the warping constant and (iv) I_ψ^* and $I_{\omega\psi}^*$ are null. Note that this specialised version of the general formulation agrees with the one derived by Attard (1986a).

3. Influence of pre-buckling deflections (I-section beams and cantilevers)

The general mathematical formulation outlined above is now used to investigate the influence of pre-buckling deflections on the LTB behaviour of prismatic and web-tapered doubly symmetric I-section simply supported beams or cantilevers.

3.1. Prismatic beams

First, one considers the cases of (i) simply supported beams acted by end moments and/or transverse forces applied at the shear centre (cross-section mid-height) and (ii) cantilevers acted by transverse forces applied at the shear centre. The corresponding governing equations, accounting or not for the effects of pre-buckling deflections, may be derived directly from Eqs. (62)-(67) and are summarised in Table 1.

Comparing the above two sets of equations and always bearing in mind that the bending moment distribution in a fundamental equilibrium state can be expressed in the form $M_y^f(\theta^1, \lambda) = \lambda f(\theta^1)$, where f defines the shape of the bending moment diagram, one is able to draw the following conclusion: whenever the transverse forces are applied at the shear centre (i.e., $z_q = z_{Ql} = 0$), the ratio M_{cr} / M_{cr}^{lin}, which relates the non-linear and linear critical moments (with and without the influence of pre-buckling deflections), is always given by

$$M_{cr}/M_{cr}^{lin} = (1 - I_z/I_y)^{-1/2} \tag{68}$$

This analytical result shows that (i) the consideration of the beam in-plane pre-buckling deflections invariably leads to a higher elastic critical moment than the one yielded by a "classical" linear stability analysis and that (ii) this increase is only relevant when the value of the ratio I_z / I_y is not too small. Moreover, since the ratio defined by Eq. (68) tends to infinity as I_z approaches I_y, one clearly confirms the well-known fact that a beam with $I_z = I_y$ experiences no LTB. It is worth noticing that Ville de Goyet (1989) reached this same conclusion for simply supported beams under uniform bending or acted by a mid-span point load. However, he used a numerical approach, based on the application of Galerkin's method (with one or two sinusoidal shape functions, thus discretising the beam into a one or two d.o.f. system), and failed to fully grasp the generality of the analytical result just presented. Furthermore, Eq. (68) also agrees very well with the closed-form solutions derived by Baker *et al.* (1956), Trahair & Woolcock (1973), Vacharajittiphan *et al.* (1974), Roberts & Azizian (1983) and Pi & Trahair (1992b), all dealing with simply supported beams under uniform bending. The slight differences that can be detected stem from the fact that the above authors included non-linear terms related to the twist, the so-called "geometric torsion". In the present formulation, the first kinematical assumption stated in 2.2 precludes the emergence of such terms.

Table 1 Governing equations for prismatic simply supported beams and cantilevers (transverse forces applied at the shear centre)

Field equilibrium equation	
Including pre-buckling deflections	Disregarding pre-buckling deflections
$EI_w\phi_{,1111} - GJ\phi_{,11} - \frac{M_y^{f2}}{EI_z}\left(1 - \frac{I_z}{I_y}\right)\phi = 0$	$EI_w\phi_{,1111} - GJ\phi_{,11} - \frac{M_y^{f2}}{EI_z}\phi = 0$
Boundary conditions	
(i) Simply supported beams	
$\phi(0) = \phi(l) = 0$	$EI_w\phi_{,11}(0) = EI_w\phi_{,11}(l) = 0$
(ii) Cantilevers	
$\phi(0) = 0 \qquad \phi_{,1}(0) = 0$	$GJ\phi_{,1}(l) - EI_w\phi_{,111}(l) = 0 \qquad EI_w\phi_{,11}(l) = 0$

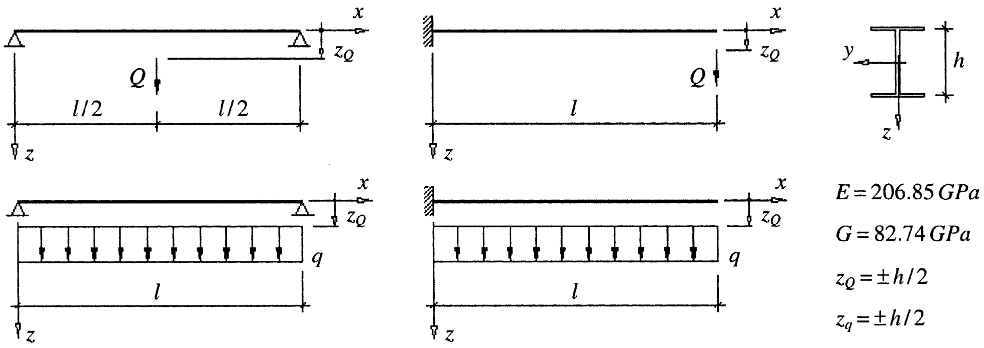

Fig. 2 Doubly symmetric prismatic I-section simply supported beams & cantilevers

When the transverse forces are not applied at the shear centre, the influence of the pre-buckling deflections on the critical moment value does not depend only on the ratio I_z / I_y, but also on (i) the location of the load point of application with respect to the shear centre (z_q or z_{Ql} value), (ii) the beam length and, to a much lesser extent, (iii) the M_y^f diagram shape. A better grasp of this assertion can be acquired by considering a set of numerical results, concerning the illustrative examples depicted in Fig. 2. These results were obtained through the application of the Rayleigh-Ritz method, using *shape* (or *coordinate*) *functions* of the form

$$\varphi_k(\theta^1) = \sin\left(\frac{k\pi}{l}\theta^1\right), k = 1, 2, \ldots \quad \text{(simply supported beams)}^8 \tag{69}$$

$$\varphi_k(\theta^1) = 1 - \cos\left[\frac{(2k-1)\pi}{2l}\theta^1\right], k = 1, 2, \ldots \quad \text{(cantilevers)} \tag{70}$$

to approximate ϕ (further details about the numerical analysis procedure can be found in Andrade 2003 and Andrade & Camotim 2003). Two different cross-section shapes were dealt with, namely the ones investigated by Trahair & Woolcock (1973):

(i) an 8UC31 section, exhibiting the following geometrical properties:
$I_y = 4566$ cm^4, $I_z = 1540$ cm^4 ($I_z / I_y = 0.337$), $J = 22.23$ cm^4, $I_w = 142.2\times10^3$ cm^6, $h = 192.2$ mm, and
(ii) a 10UB29 section, for which the relevant geometrical data are
$I_y = 6560$ cm^4, $I_z = 678.5$ cm^4 ($I_z / I_y = 0.103$), $J = 25.72$ cm^4, $I_w = 103.4\times10^3$ cm^6, $h = 246.9$ mm.

The loads were applied at the top and bottom flanges (z_q, $z_Q = \pm h/2$). For the simply supported beams, it was found that accurate estimates of the critical moments M_{cr}^{lin} and M_{cr} could be achieved with only five shape functions. The cantilevers, however, required the use of up to nine shape functions and, moreover, it was observed that convergence is somewhat slower for the longer cantilevers with bottom flange loading.

Figs. 3-4 show the variation of the critical moment percentage increase due to the pre-buckling deflections, $(M_{cr}-M_{cr}^{lin})/ M_{cr}^{lin}$, with the beam/cantilever length, for 4.0 m $\leq l \leq$ 12.0 m (simply supported

[8] It is worth pointing out that, due to symmetry, only the odd-number shape functions were considered in the analysis of the simply supported beams shown in Fig. 2.

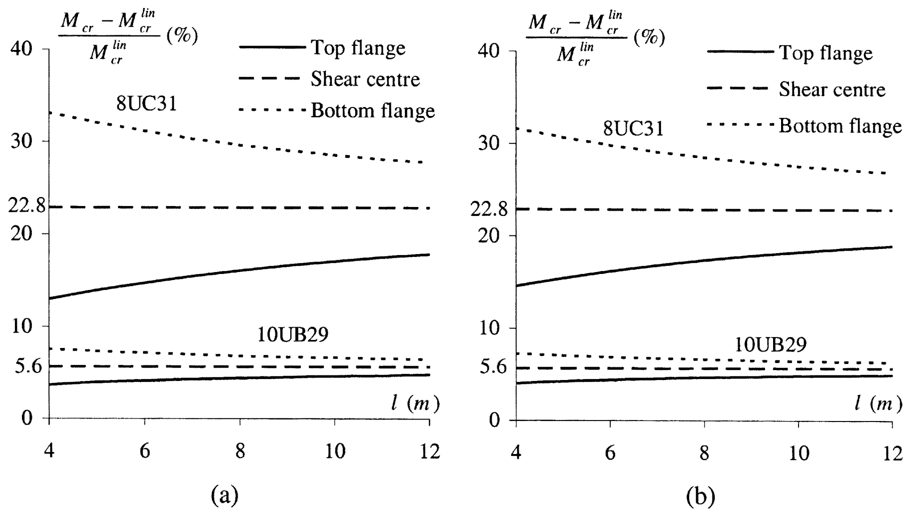

Fig. 3 Effect of pre-buckling deflections on the M_{cr} values of simply supported beams acted by (a) mid-span point loads and (b) uniformly distributed loads

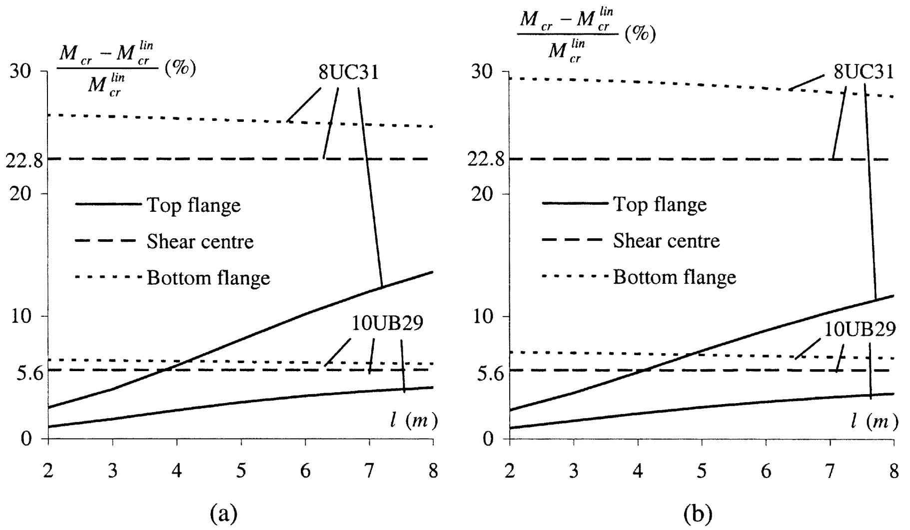

Fig. 4 Effect of pre-buckling deflections on the M_{cr} values of cantilevers acted by (a) tip loads and (b) uniformly distributed loads

beams) or 2.0 m $\leq l \leq$ 8.0 m (cantilevers). The observation of these curves prompts the following remarks:

(i) Concerning the simply supported beams (Fig. 3), the largest increase occurs for the shortest and stockiest beam subjected to a bottom flange mid-span point load. The curves associated with bottom (top) flange loading always lie above (below) the horizontal lines yielded by Eq. (68) and valid for shear centre loading. As the beam length grows, these horizontal lines are approached in an almost symmetrical way.

(ii) As far as the cantilevers are concerned (Fig. 4), one notices that the critical moment increase is invariably larger (smaller) for bottom (top) flange loading than for shear centre loading. Moreover, while (ii_1) the critical moment increase for bottom flange loading is only slightly dependent on the beam length, (ii_2) the influence of the beam length is much more pronounced for top flange loading.

(iii) Some of the above LTB problems were also solved by Trahair & Woolcock (1973), Vacharajittiphan *et al.* (1974), Attard (1986b) and Ronagh *et al.* (2000b). In all cases, the results presented here fully agree with the values reported by these authors.

The influence of pre-buckling deflections on LTB is closely related to $W^f_{,11}(\theta^1, \lambda) = -M^f_y/(EI_y)$, as can be seen from Eq. (39) or Eq. (61). For a given beam and loading profile (defining the shape of the $W^f_{,11}$ diagram), the critical moment increase associated with the pre-buckling deflections grows with λ_{cr} (defining the magnitude of the $W^f_{,11}$ diagram at buckling). This explains why the ratio $(M_{cr} - M^{lin}_{cr})/M^{lin}_{cr}$ grows monotonically with $z_{q(Q)}$ (all other factors being equal).

For the simply supported beams analysed, accurate M^{lin}_{cr} predictions can be obtained by using the Eurocode 3 formula (CEN 1992)

$$M^{lin}_{cr} = C_1 \frac{\pi^2 EI_z}{l^2}\left(\sqrt{\frac{I_w}{I_z} + \frac{l^2 GJ}{\pi^2 EI_z} + (C_2 z_{q(Q)})^2} + C_2 z_{q(Q)}\right) \tag{71}$$

with $C_1 = 1.363$ and $C_2 = 0.553$ (mid-span point loads) or $C_1 = 1.132$ and $C_2 = 0.459$ (uniformly distributed loads). Following a suggestion made by Mohri *et al.* (2002), it was found that this formula also yields accurate M_{cr} predictions when the pre-buckling deflections are accounted for, provided that the above C_1 and C_2 values are replaced by the modified coefficients

$$C^{mod}_1 = C_1(1 - I_z/I_y)^{-1/2} \qquad C^{mod}_2 = C_2(1 - I_z/I_y)^{-1/2} \tag{72}$$

3.2. Tapered beams

In this sub-section, several numerical results concerning the LTB behaviour of doubly symmetric web-tapered I-section simply supported beams and cantilevers are presented and discussed. They make it possible to assess how the critical moment increase due to the pre-buckling deflections is affected by a number a factors, namely (i) the web height tapering ratio, (ii) the location of the load point of application, (iii) the beam length, (iv) the M^f_y diagram shape and (v) the support conditions. These results were obtained by means of the Rayleigh-Ritz method, using the shape functions given in Eqs. (69) and (70), and concern the beams and cantilevers shown in Fig. 5, which exhibit the following two sets of geometrical and material properties:

(i) $E = 200$ GPa, $G = 77.2$ GPa, $b = 152.4$ mm, $h_{max} = 609.6$ mm, $t_f = 12.7$ mm, $t_w = 9.5$ mm, $l = 6096$ mm and variable tapering ratio $\alpha = h_{min} / h_{max}$ - Data I.

(ii) $E = 206.85$ GPa, $G = 82.74$ GPa, $b = 150$ mm, $h_{max} = 300$ mm, $t_f = 10$ mm, $t_w = 6$ mm, $\alpha = 0.5$ and variable length l - Data II.

The curve displayed in Fig. 6(a) concerns Beam A + Data I and corresponds to a point load applied at the mid-section centroid (i.e., $z_Q = 0$). It provides the variation of Q^{lin}_{cr} (pre-buckling deflections neglected) with the tapering ratio $\alpha = h_{\min} / h_{\max}$, which is comprised between $\alpha = 0.2$ and $\alpha = 1.0$

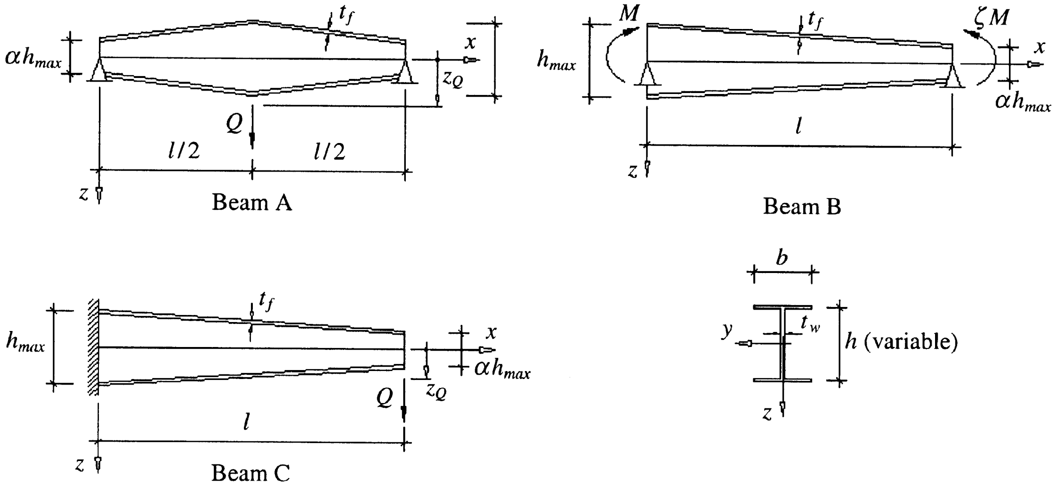

Fig. 5 Doubly symmetric web-tapered I-section simply supported beams & cantilevers

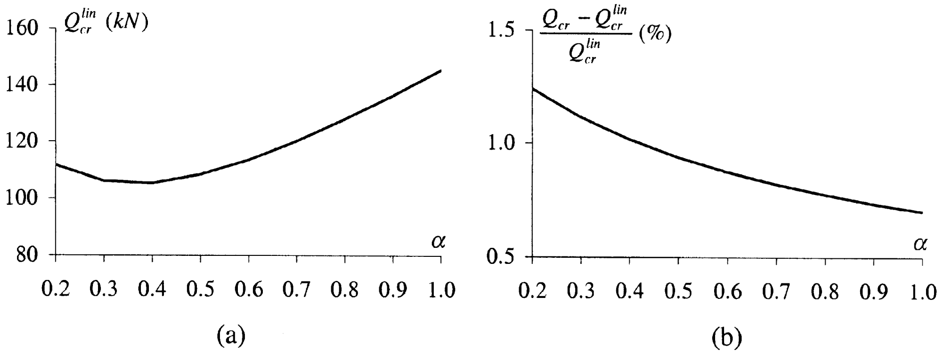

Fig. 6 Beam A + Data I: (a) variation of Q_{cr}^{lin} with a and (b) effect of pre-buckling deflections

(prismatic beam). The number of shape functions required to achieve convergence varied from two (for $\alpha = 1.0$) to twelve (for $\alpha = 0.2$) - recall that, due to symmetry, only the odd-number shape functions need to be considered in Eq. (69). These Q_{cr}^{lin} values are in close agreement with the results reported earlier by Yang & Yau (1987) and, more recently, by Boissonnade & Muzeau (2001). Fig. 6(b), on the other hand, provides the variation, with α, of the critical load increase $(Q_{cr} - Q_{cr}^{lin})\,/Q_{cr}^{lin}$ due to the pre-buckling deflections (the same number of shape functions was used to evaluate both Q_{cr}^{lin} and Q_{cr}). One observes that this increase (i) is always quite small (it never exceeds 1.25%) and (ii) progressively decreases as α increases, most likely because the average I_z^*/I_y^* value also decreases. Indeed, while I_z^* does not vary (the flanges are uniform), the average I_y^* value increases with a. For example, when $\alpha = 0.2$, the ratio I_z^*/I_y^* varies between 0.014 (mid-span) and 0.473 (end sections), while the associated prismatic beam ($\alpha = 1$) exhibits a constant ratio of 0.014. Note, however, that the maximum bending moment and buckling mode curvature occur at mid-span, where the I_z^*/I_y^* ratio remains constant.

In order to assess how the relevance of the pre-buckling deflections varies with the (i) point of load application and (ii) beam length, one analyses Beam A + Data II. The beam lengths considered range from 5 m to 12 m and the point load is applied at three different locations, namely at the mid-span

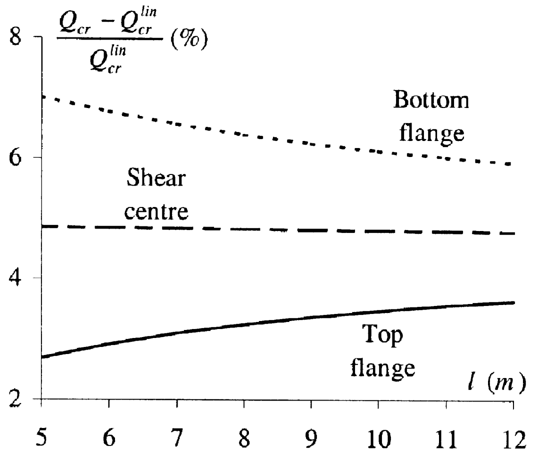

Fig. 7 Beam A + Data II: variation of critical load increase with beam length and point of load application

cross-section (i) mid-height (shear centre), (ii) top flange and (iii) bottom flange. The inertia ratio I_z^*/I_y^* varies between 0.069 (mid-span) and 0.303 (end sections). The computation of Q_{cr}^{lin} and Q_{cr} required the use of up to ten (odd-number) shape functions given by Eq. (69) - a larger number of shape functions was necessary for the smaller lengths, which are associated with steeper flange slopes. The curves displayed in Fig. 7 provide the critical load percentage increase due to the pre-buckling deflections. These curves show that, as the beam length increases, both the $(Q_{cr} - Q_{cr}^{lin})/Q_{cr}^{lin}$ ratios concerning top and bottom flange loading approach the (almost constant) value associated with shear centre loading (≈5%). Furthermore, the critical load increase (i) is virtually independent of the beam length, for shear centre loading, and (ii) mildly depends on l, for bottom or top flange loading. One should still point out that the results given in Fig. 7 are in excellent agreement with the values reported by Ronagh *et al.* (2000b) - to the authors' best knowledge, these are the only currently available results dealing with the influence of the pre-buckling deflections on the LTB behaviour of tapered beams.

Next, one analyses Beam B + Data II, in order to assess the influence of the bending moment diagram shape. Three linear diagrams are dealt with (ζ= 1.0, 0.5, 0 - see Fig. 5) and a maximum of eight shape functions given by Eq. (69) were used to compute both M_{cr}^{lin} and M_{cr}. The ensuing $(M_{cr} - M_{cr}^{lin})/M_{cr}^{lin}$ vs. l. curves are almost parallel, as shown in Fig. 8. The influence of the pre-buckling deflections increases with ζ (at a decreasing rate, though), most likely because larger bending moments are acting on the beam segment with the highest I_z^*/I_y^* ratio, i.e., in the vicinity of the tapered end (note that the variation of the I_z^*/I_y^* value is the same as the one occurring in half of the simply supported beam analysed in the previous problem).

Finally, one considers the web-tapered cantilever shown in Fig. 5 (Beam C), together with Data II. The length l ranges from 3 m to 8 m and, as before, the point load Q is applied at the free end section (i) mid-height (shear centre), (ii) top flange and (iii) bottom flange. The numerical results obtained (considering nine shape functions given by Eq. (70)) are displayed in Fig. 9 and indicate that the critical load percentage increase due to the pre-buckling effects is almost independent of l, for both shear centre and bottom flange loading (a higher value is observed in the latter case). For top flange loading, on the other hand, the ratio $(Q_{cr} - Q_{cr}^{lin})/Q_{cr}^{lin}$ increases significantly with l, progressively approaching the curve related to shear centre loading.

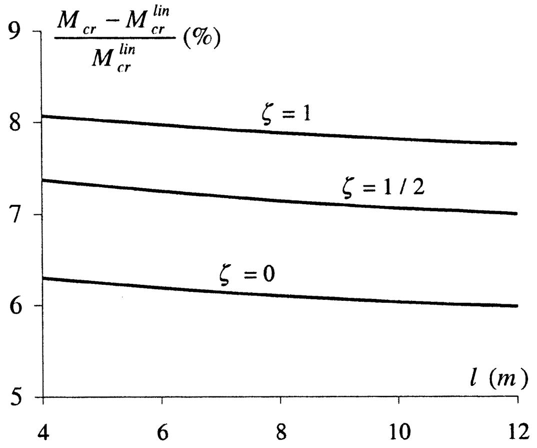

Fig. 8 Beam B+Data II: variation of critical moment increase with the beam length and bending moment diagram shape

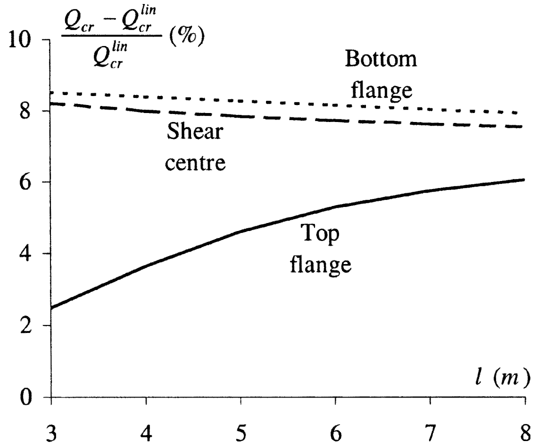

Fig. 9 Beam C+Data II: variation of $(Q_{cr}-Q_{cr}^{lin}) / Q_{cr}^{lin}$ with cantilever length and load point of application

4. Conclusions

A unified variational approach to analyse the lateral-torsional buckling (LTB) behaviour of doubly symmetric prismatic and tapered beams, taking into account the influence of the pre-buckling deflections, was presented. It extends previous work developed by the authors, concerning the *linear* stability behaviour of this type of beams (Andrade & Camotim 2003). This formulation was then applied to simply supported beams and cantilevers, and, in this more restricted context, it was demonstrated that both the weak and strong statements of the LTB problem can be expressed in terms of a *single* unknown function, namely the torsional rotation ϕ.

The remainder of the paper was devoted to investigating the influence of the pre-buckling deflections on the LTB behaviour of prismatic and web-tapered I-section simply supported beams and cantilevers. First, an analytical result was established, concerning prismatic members acted by end moments and transverse loads applied at the shear centre. Then, several elastic critical moments or loads, obtained by means of the Rayleigh-Ritz method (with trigonometric shape functions), were presented, discussed and, when possible, also compared with values reported in the literature. These numerical results made it possible to illustrate and somewhat quantify (i) the influence of the pre-buckling deflections on the LTB behaviour of prismatic and tapered beams, (ii) the qualitative differences existing between the LTB behaviours of simply supported beams and cantilevers and also (iii) how such behaviour is affected by several factors, namely the (iii_1) minor-to-major inertia ratio, (iii_2) beam length, (iii_3) position of the load point of application and (iii_4) bending moment diagram shape.

Finally, one last word to mention, once more, that this paper only dealt with doubly symmetric beams, a restriction exclusively due to space limitations. Indeed, the authors have already developed and numerically implemented a more general version of the formulation presented here, which is valid for singly symmetric beams loaded in their symmetry plane (Andrade & Camotim 2002, Andrade 2003). Hopefully, applications of this more general formulation will be reported in the near future.

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