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The analytic solution for parametrically excited oscillators of complex variable in nonlinear dynamic systems under harmonic loading

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Abstract. In this paper we have considered the vibration of parametrically excited oscillator with strong cubic positive nonlinearity of complex variable in nonlinear dynamic systems with forcing based on Mathieu-Duffing equation. A new analytical approach called homotopy perturbation has been utilized to obtain the analytical solution for the problem. Runge-Kutta's algorithm is also presented as our numerical solution. Some comparisons between the results obtained by the homotopy perturbation method and Runge-Kutta algorithm are shown to show the accuracy of the proposed method. In has been indicated that the homotopy perturbation shows an excellent approximations comparing the numerical one.

Keywords: Homotopy Perturbation Method (HPM); Runge-Kutta Method (RKM); parametrically excited oscillator

1. Introduction

Nonlinear phenomena occurs in engineering and physical sciences. One of the main interests in nonlinear science is the study on these nonlinear problems analytically. We have difficulty for finding an exact solution for these nonlinear problems and they have to be solved with other approximate analytical methods.

Perturbation technique is one of the well-known analytical methods. They are not practical for strongly nonlinear equations, so for conquer the imperfections, novel techniques have been appeared in open literature, for instance: (He 1999a), Hamiltonian approach (Bayat and Pakar 2011a, 2012, 2013a, Bayat *et al*. 2014a, b), energy balance method (He 2002, Bayat and Pakar 2011b, Pakar and Bayat 2011, 2012, Mehdipour *et al*. 2010), variational iteration method (Dehghan and Tatari 2010, Pakar *et al*. 2012, He 1999b), amplitude frequency formulation (Bayat *et al*. 2011, 2012, Pakar and Bayat 2013a, He 2008), max-min approach (Shen and Mo 2009, Zeng and Lee 2009, Pakar and Bayat 2011, 2012), variational approach method (He 2007, Bayat and Pakar 2013a, Bayat *et al*. 2011, 2013), and the other approximare methods (Xu and Zhang 2009,

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Xu 2008, Shaban *et al*. 2010, Kuo and Lo 2009, Wu 2011, Odibat *et al*. 2008, Filobello-Nino *et al*. 2012, Ganji 2006, Ağırseven and Öziş 2010, Bayat and Abdollahzade 2011c, d, Bayat *et al*. 2014a, Andrianov *et al*. 2004).

Among these methods, homotopy perturbation method is considered to analysis the vibration of parametrically excited oscillator with strong cubic positive nonlinearity of complex variable in nonlinear dynamic systems with forcing based on Mathieu-Duffing equation.

Applications of homotopy perturbation method have been studied, to demonstrate the applicability and preciseness of the method, some comparisons between analytical and numerical solutions are presented. Eventually we have shown that HPM can converge to a precise cyclic solution for nonlinear equations.

The governing equation of Mathieu-Duffing system which is considered in this study is described by the following high-order nonlinear differential equation (Andrianov *et al*. 2004)

$$\ddot{x} + \gamma\dot{x} + \left[\xi + 2\varepsilon\cos(2t)\right]x + \beta x^3 = F_0 \sin 2t, \quad x(0) = 1, \quad \dot{x}(0) = 0 \tag{1}$$

Where dots indicate differentiation with respect to the time (t), $\varepsilon << 1$ is a small parameter, β is the parameter of nonlinearity, and ξ is the transient curve.

2. The basic concept of the solutions

In this section, the basic of the utilized methods are explained for the better understanding of the reader.

2.1 Basic idea of HPM

To illustrate the basic ideas of this method, we consider the following equation

$$A(x) - f(r) = 0 \quad r \in \Omega \tag{2}$$

with the boundary condition of

$$B\left(x, \frac{\partial x}{\partial t}\right) = 0 \quad r \in \Gamma \tag{3}$$

Where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω. A can be divided into two parts of L and N, where L is linear and N is nonlinear. Eq. (2) can therefore be rewritten as follows

$$L(x) + N(x) - f(r) = 0 \quad r \in \Omega \tag{4}$$

Homotopy perturbation structure is shown as follows

$$H(v, p) = (1-p)\left[L(v) - L(x_0)\right] + p\left[A(v) - f(r)\right] = 0 \tag{5}$$

where

$$v(r, p): \Omega \times [0,1] \to R \tag{6}$$

In Eq. (5), $p \in [0,1]$ is an embedding parameter and x_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. (2) can be written as a power series in p, as following

$$v = v_0 + pv_1 + p^2 v_2 + \cdots = \sum_{i=0}^{n} v_i p^i \tag{7}$$

And the best approximation for the solution is

$$x = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{8}$$

2.2 Basic idea of Runge-Kutta

The fourth RK (Runge-Kutta) method has been used to verify the homotopy perturbation solution.

This iterative algorithm is written in the form of the following formulae for the second-order differential equation

$$\begin{aligned} \dot{x}_{i+1} &= \dot{x}_i + \frac{\Delta t}{6}\left(h_1 + 2h_2 + 2h_3 + k_4\right) \\ x_{i+1} &= x_i + \Delta t\left(\dot{x}_i + \frac{\Delta t}{6}(h_1 + h_2 + h_3)\right) \end{aligned} \tag{9}$$

Where, Δt is the increment of the time and h_1, h_2, h_3, and h_4 are determined from the following formulae

$$\begin{aligned} h_1 &= f(\dot{x}, x_i, \dot{x}_i)k, \\ h_2 &= f\left(t_i + \frac{\Delta t}{2}, x_i + \frac{\Delta t}{2}\dot{x}_i, \dot{x}_i + \frac{\Delta t}{2}h_1\right), \\ h_3 &= f\left(t_i + \frac{\Delta t}{2}, x_i + \frac{\Delta t}{2}\dot{x}_i, \frac{1}{4}\Delta t^2 h_1, \dot{x}_i + \frac{\Delta t}{2}h_2\right), \\ h_4 &= f\left(t_i + \Delta t, x_i + \Delta t\,\dot{x}_i, \frac{1}{2}\Delta t^2 h_2, \dot{x}_i + \Delta t\, h_3\right). \end{aligned} \tag{10}$$

The numerical solution starts from the boundary at the initial time, where the first value of the displacement function and its first-order derivative are determined from initial condition. Then, with a small time increment Δt, the displacement function and its first-order derivative at the new position can be obtained using Eq. (9). This process continues to the end of the time limit.

3. The solutions

In this section the applications of the two methods to the nonlinear equation of oscillator are discussed.

3.1 homotopy perturbation method

As the HPM was applied to the nonlinear equation of (1), we have

$$(1-p)\left(\ddot{x}+\gamma\dot{x}+\xi x\right)+p\left(\ddot{x}+\gamma\dot{x}+\left[\xi+2\varepsilon\cos(2t)\right]x+\beta x^{3}-F_{0}\sin 2t\right)=0 \tag{11}$$

After expanding the equation and collecting it based on the coefficients of p-terms, we have

$$p^{0}:\quad \ddot{x}_0+\gamma\dot{x}_0+\varepsilon x_0=0 \tag{12}$$

$$p^{1}:\quad \ddot{x}_1+\gamma\dot{x}_1+\xi_1 x_1-2\varepsilon x_0+\beta x_0^{3}+4\varepsilon\cos^{2}(t)x_0-F_0\sin(2t)=0 \tag{13}$$

$$p^{2}:\quad \ddot{x}_2+\gamma\dot{x}_2+\xi x_2-2\varepsilon x_1+4\varepsilon\cos^{2}(t)x_1+3\beta x_0 x_1^{2}=0 \tag{14}$$

$$p^{3}:\quad \ldots\ldots \tag{15}$$

One can now try to obtain the solution of different iterations (12), in the form of

$$x_0(t)=\frac{1}{2}\frac{\left(\gamma+\sqrt{\gamma^{2}-4\xi}\right)e^{\left(-\frac{1}{2}\gamma+\frac{1}{2}\sqrt{\gamma^{2}-4\xi}\right)t}}{\sqrt{\gamma^{2}-4\xi}}-\frac{1}{2}\frac{\left(\gamma-\sqrt{\gamma^{2}-4\xi}\right)e^{\left(-\frac{1}{2}\gamma-\frac{1}{2}\sqrt{\gamma^{2}-4\xi}\right)t}}{\sqrt{\gamma^{2}-4\xi}} \tag{16}$$

The obtained iteration is used to generate the equation for the next iteration, and therefore the second and third iterations are obtained. Since the two other ones and therefore the general solution are too long to be written in this article, we have shown them in figures.

3.2 Runge-Kutta (Numerical)

In this section, the Maple Package has been utilized for the numerical analysis of the problem, in which the rkf45 is used to solve ODEs. The solution for the displacement and the velocity for eleven different points of time are shown in Table 1.

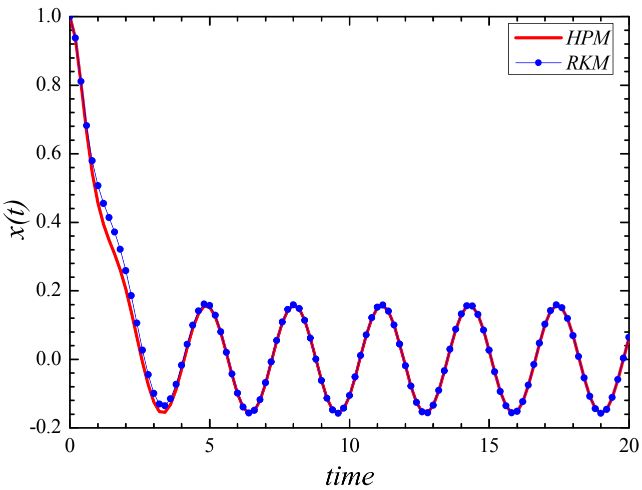

Fig. 1 Comparison of displacement time history for Case 1 : $\gamma = 3$, $\xi = 2$, $\varepsilon = 0.01$, $\beta = 2$, $F_0 = 1$

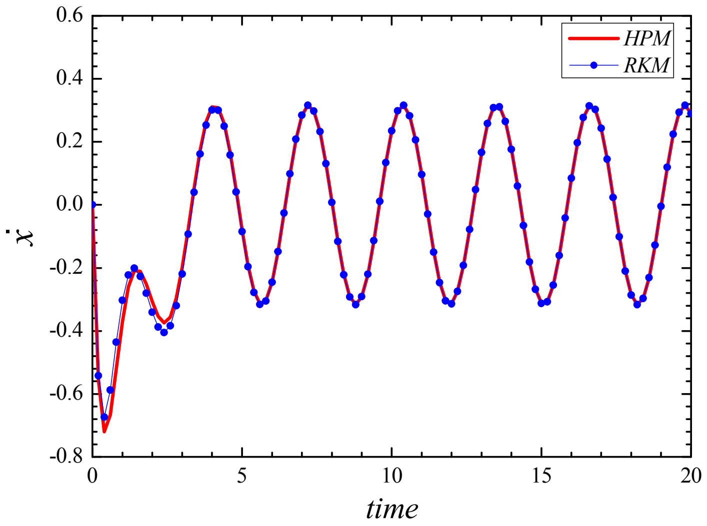

Fig. 2 Comparison of time history diagram of velocity $\dot{x}$ for Case 1 : $\gamma = 3$, $\xi = 2$, $\varepsilon = 0.01$, $\beta = 2$, $F_0 = 1$

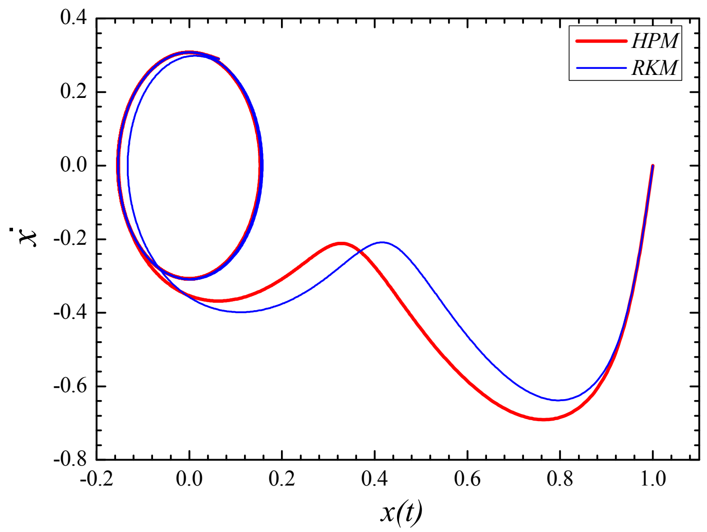

Fig. 3 Comparison of velocity $\dot{x}$ based on displacement x for Case 1 : $\gamma = 3$, $\xi = 2$, $\varepsilon = 0.01$, $\beta = 2$, $F_0 = 1$

4. Results and discussions

In this section, to illustrate and verify the accuracy of this new approximate analytical approach, a comparison between homotopy perturbation method and numerical ones are presented in Figs. 1 to 6 for Mathieu-Duffing equation. The Figs. 1 to 3 for one case and Figs. 4 to 6 for another case.

Figs. 1 and 4 are the displacement time history comparisons and the Figs. 2 and 5 are the velocity time history comparisons of the problem. From those figures, the motion of the system is a periodic motion and the amplitude of vibration is a function of the initial conditions.

Comparisons of homotopy perturbation method for different parameters via numerical is presented in Table1.

Table 1 is also presented to compare the point value of eleven extreme points of homotopy perturbation method and Runge-Kutta's algorithm. An excellent agreement can be seen in Table 1 between the analytical method and numerical one.

To further illustration and verification of the proposed method, some comparison of homotopy

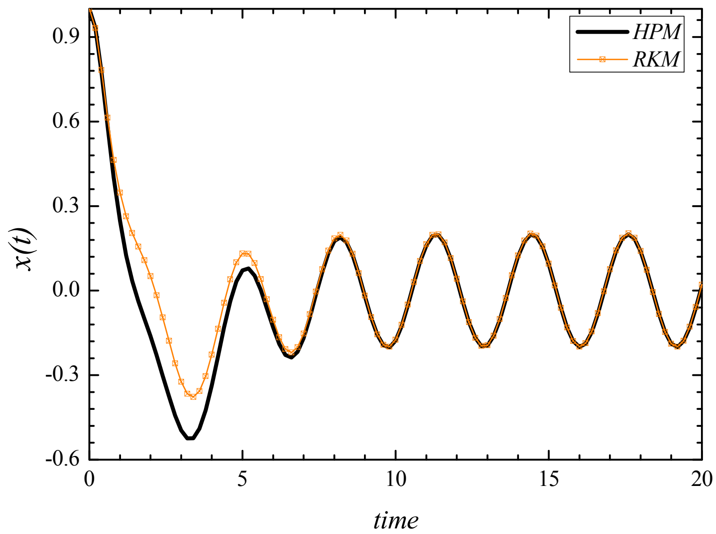

Fig. 4 Comparison of time history diagram of displacement $x(t)$ for Case 2 : $\gamma = 2$, $\xi = 1$, $\varepsilon = 0.03$, $\beta = 3$, $F_0 = 1$

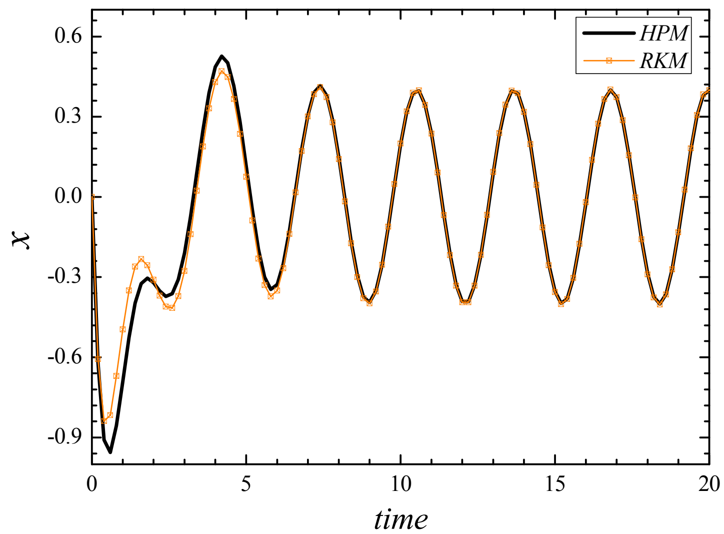

Fig. 5 Comparison of time history diagram of velocity $\dot{x}$ for Case 2 : $\gamma = 2$, $\xi = 1$, $\varepsilon = 0.03$, $\beta = 3$, $F_0 = 1$

perturbation method and numerical solution are presented in Figs. 3 and 6 for $x(t)$ and $\dot{x}(t)$ for different parameters of the systems.

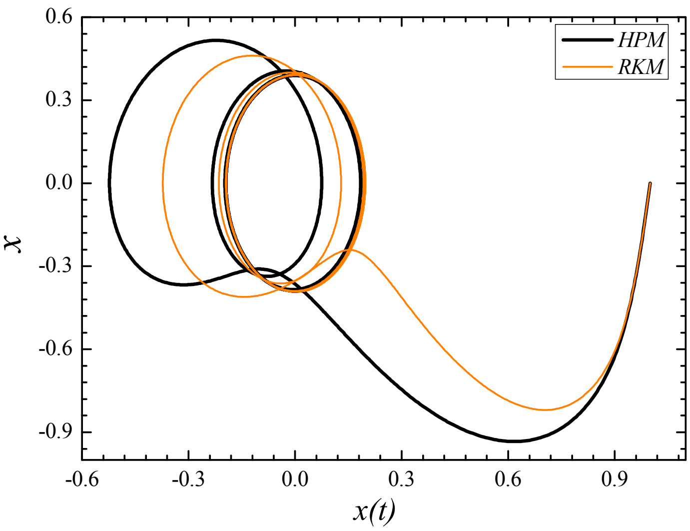

Fig. 6 Comparison of velocity $\dot{x}$ based on displacement x for Case 2 : $\gamma = 2$, $\xi = 1$, $\varepsilon = 0.03$, $\beta = 3$, $F_0 = 1$

Table 1 Comparative table for error detection of the analytic method, for $\gamma = 3$, $\xi = 2$, $\varepsilon = 0.01$, $\beta = 2$, $F_0 = 1$

Time	Displacement (x)			Velocity ($\dot{x}$)		
	HPM	RKM	Error (%)	HPM	RKM	Error (%)
0	1	1	0	0	0	0
5	0.152	0.157	2.993	-0.081	-0.084	3.489
6	-0.100	-0.099	1.177	-0.245	-0.246	0.489
7	-0.070	-0.068	2.936	0.284	0.284	0.203
8	0.158	0.159	0.765	0.009	0.008	2.314
9	-0.062	-0.062	0.203	-0.291	-0.292	0.092
10	-0.107	-0.106	1.265	0.233	0.235	0.769
11	0.150	0.152	0.830	0.097	0.096	1.313
12	-0.018	-0.018	0.707	-0.314	-0.315	0.157
13	-0.135	-0.134	0.756	0.164	0.166	1.426
14	0.131	0.132	1.061	0.178	0.177	0.510
15	0.026	0.026	0.167	-0.312	-0.313	0.251
16	-0.153	-0.152	0.435	0.082	0.085	2.972
17	0.101	0.102	1.476	0.244	0.243	0.265
18	0.069	0.069	0.134	-0.285	-0.286	0.402
19	-0.158	-0.158	0.205	-0.007	-0.006	2.989
20	0.063	0.064	2.467	0.290	0.290	0.153

As shown in Figs. 3 and 6 and Table 1, it is apparent that homotopy perturbation method has an excellent agreement with the numerical solution using Rung-Kutta and these expressions are valid for a wide range.

5. Conclusions

In this study the homotopy perturbation method has been employed to analyze of parametrically excited oscillator with strong cubic positive nonlinearity of complex variable in nonlinear dynamic systems with forcing. The results obtained from this method have been compared with those obtained from numerical method using RK algorithm. This comparison shows excellent agreement between the two methods. Also, HPM does not require small parameters, so the limitation of the conventional perturbation method could be eliminated. The results indicated that HPM is extremely speedy, light, with high accuracy. Excellent agreement between approximate solution and the numerical one is demonstrated and discussed. The method can be easily extended to any nonlinear oscillator without any difficulty.

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