Computers and Concrete, Vol. 9, No. 3 (2012) 227-00
DOI: http://dx.doi.org/10.12989/cac.2012.9.3.227

Generalization of shear truss model to the case of SFRC beams with stirrups

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(Received September 23, 2010, Revised April 30, 2011, Accepted May 2, 2011)

Abstract. A theoretical model for shear strength evaluation of fibrous concrete beams reinforced with stirrups is proposed. The formulation is founded on the theory of plasticity and the stress field concepts, generalizing a known plastic model for calculating the bearing capacity of reinforced concrete beams, to the case of fibrous concrete. The beneficial effect of steel fibres is estimated taking into account the residual tensile strength of fibrous concrete, by modifying an analytical constitutive law which presents a plastic plateau as a post-peak branch. Around fifty results of experimental tests carried out on steel fibrous concrete beams available in the literature were collected, and a comparison of shear strength estimation provided by other semi-empirical models is performed, proving that the numerical values obtained with the proposed model are in very good agreement with the experimental results.

Keywords: eurocode 2; shear; steel fibre reinforced concrete; stirrup; design.

1. Introduction

The behaviour of reinforced concrete (RC) structural elements, predominantly subjected to shear actions, remains an area of much concern. The complexity of the problem arises from the presence of many different physical phenomena, such as the propagation of cracks and the compression stress transfer mechanism from loading points to supports. Moreover, geometry, load conditions and arrangement of longitudinal and transversal reinforcement influence the response of the structural member. Thus design codes are continually revised and generally become more stringent, suggesting design formulations which are usually derived from theoretical models in order to provide a wide range of use. However, analysis and design methods still have to be improved to provide a more rational tool for the prediction of shear strength. To this aim, only formulations based on a robust theoretical approach allow accurate estimation of shear strength in a wide range of cases.

The use of steel fibre reinforced concrete (SFRC) has been growing for the last two decades, particularly when crack propagation control is of primary importance (Muttoni and Ruiz 2008). Many authors (e.g. Casanova and Rossi 1997, Spinella *et al.* 2010) have reported that the addition of steel fibres results in significant improvement of the shear capacity for reinforced beams, and a large number of experimental results are available concerning beams with longitudinal reinforcement for flexure and steel fibres only for shear (Adebar *et al.* 1997). Moreover, it has been shown (Cucchiara *et al.* 2004) that steel fibres can partially substitute shear steel reinforcements (generally constituted by vertical stirrups and horizontal steel bars), obtaining equivalent performance in terms of both

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strength and ductility. The use of transversal reinforcement made up of both steel fibres and stirrups seems to be a promising application. As a matter of fact, the post-cracking stress in tension due to fibres is not always sufficient to totally replace the steel reinforcement, and at the same time the placement of a large amount of stirrups can result in reinforcement that is so dense as to compromise the cast phase. By contrast, an optimized mix of steel fibres and stirrups as shear reinforcement assures a ductile response of structural elements, which is of practical importance in the design of structural members in seismic regions. In this field, the capacity of absorbing the energy due to seismic actions is nowadays required in widely adopted design codes, incorporating performance-based criteria and advanced tools for numerical analysis (i.e. push-over).

The design formula for shear strength incorporated in most international building codes of practice derives from robust theoretical approaches, such as: the Modified Compression Field Theory (MCFT) (Vecchio and Collins 1986) for the Canadian Code (CSA A23.3-04 2004); the AASHTO LRFD8 (2007), the rational model based on the Critical Shear Crack Theory (CSCT) (Muttoni and Ruiz 2008) and adopted by the Swiss Code (SIA 262 2003). In Eurocode 2 (2004) (EC2) the classical model of Ritter and Mörsch based on the truss analogy (with a 45 degree angle for the diagonal compression strut) is replaced by a more general model based on the plastic theory, which takes stress distributions into account as uniform inclined fields (Nielsen 1999). The plastic approach assumes that the angle θ, which defines the inclination of the compressive stress field with respect to the horizontal axis, is not fixed at 45 degrees, but may be variable in a wide range of values. This theoretical model has been enhanced to evaluate *N-M-V* interaction resistance domains for box and I-shaped reinforced concrete cross-sections (Recupero *et al.* 2003).

In this paper, the plastic model originally formulated by Nielsen *et al.* (1999), which is the basis of the EC2 (2004) shear design formulation, is extended to the case of structural elements reinforced in shear with both steel fibres and stirrups. The ability of fibres to provide residual tensile strength of the composite is mainly taken into account rearranging an appropriate theoretical constitutive law for SFRC (Lim *et al.* 1987); thus the ability of material to transfer tensile stress across cracks at the ultimate limit state is taken into account by the plastic model theory. Furthermore, flattened stress-strain constitutive laws in the post-peak range of fibrous concrete behaviour in tension make the SFRC more suitable than plain concrete for the application of the plastic theory. A stress field corresponding to longitudinal reinforcement along the web is also included in the proposed model (Mancini *et al.* 1996).

A database of test results (Swamy and Bahia 1985, Furlan and Bento de Hanai 1997, Campione and Mindess 1999, Dupont and Vandewalle 2003, Campione *et al.* 2004, Kearlsley and Mostert 2004) reported in the literature was collected and two well-known semi-empirical models are considered (Thomas and Ramaswamy 2006, Campione *et al.* 2004). The comparisons between the shear strength estimation provided by the proposed model and the analytical formulations taken into account highlight the effectiveness of the proposed theoretical model to evaluate the shear strength of beams reinforced with steel fibres and stirrups.

Then a numerical analysis is performed to investigate the effectiveness of the proposed analytical model in predicting the ability of fibres to shift the beam failure mode from brittle for shear to ductile for flexure.

2. A new shear strength model for sfrc beams with stirrups

Since the beginning of the 20th century, when Ritter and Mörsch postulated the earliest truss models, great progress has been made in the investigation of the analytical solution of shear problems in RC. However, most of the proposed highly sophisticated tools (Vecchio and Collins 1986) require considerable simplification to make them suitable for codes of practice. Nonetheless, new theoretical concepts have been progressively introduced, such as the idealization of the stress field, certainly performing better in describing the behaviour of the structural member than the classical strut-and-tie model.

The newly developed models have strongly influenced international design codes for RC elements (CSA 2004, EC2 2004), and the truss-based model formulation was rapidly replaced with analytical expressions derived from the stress field assumption.

2.1 Stress field model

The proposed shear strength model is based on the work of Nielsen *et al.* (1999), where compression stress fields with variable inclination are used to evaluate the shear capacity of RC elements with stirrups. Nielsen's model is founded on the plastic theory, and it is derived on the basis of the theorems of limit analysis, more precisely the static theorem, which provide the actual shear strength of the beam as the maximum shear among those of the statically admissible solutions (lower bound solution). Each statically admissible solution can be derived from the equilibrium condition of a free body diagram in which the internal stresses satisfy the plastic admissibility condition; the failure mechanism is then determined, depending on which stress field reaches its ultimate stress. In order to derive the model, the following assumptions are made:

- The distribution of internal force in the beam at failure is described by the free body diagram in Fig. 1, where two internal forces are considered, namely those acting on the top and bottom chords, and two stress fields due to the concrete of the web and the web reinforcement.
- The top and bottom chords are dimensionless and subjected to axial force only, i.e. have been idealized as stringers carrying concentrated internal forces (Nielsen 1999) made up of compressed concrete and longitudinal reinforcement, respectively
- The compression concrete web stress field is assumed to be uniform as provided by the theory of plasticity, and it is inclined by an angle θ on the longitudinal axis, which may differ from 45°
- Stirrups, placed inclined by an angle a on the longitudinal direction, are assumed to be smeared along the depth of the beam and able to transfer axial force only.

Further shear strength mechanisms, such as dowel action and aggregate interlock, are only apparently neglected. They are taken into account by means of the inclination of the compression web stress field, which is different from that of the crack. This physical behaviour plays a key role in the shear failure mechanism at the basis of the model; as a matter of fact shear slips along cracks deteriorate the effective strength of the compressed concrete (Nielsen 1999, Zhang 1997, Vecchio 2000), which is limited to values lower than the capacity of the material measured by a uniaxial compression test.

An interesting extension of Nielsen's theory was provided by Mancini *et al.* (1996) for the case of RC beams with surface reinforcement. They generalised Nielsen's model by introducing another horizontal stress field to represent the action of skin reinforcement when it is present.

The presence of longitudinal reinforcement along the depth of a member is relevant for shear behaviour. Kuchma *et al.* (1997) demonstrated the highly beneficial effect on shear strength provided by the addition of three longitudinal layers of relatively low diameter bars along the depth of a specimen 1000 mm deep. As a result, the shear strength of the structural element increased about

50%, the ductility was doubled and a well distributed crack pattern was formed, without any early localization of shear-critical cracks.

The addition of fibres is able to providing a similar effect in-member response and, as will be seen later, the formulation proposed here for SFRC beams includes a simple term to take into account the contribution of fibres to shear strength. Fibres help the concrete matrix to contain the shear slips along the crack surface (Spinella *et al.* 2010). Consequently, they reduce the deterioration of concrete, allowing higher effective compression strength for fibrous concrete than for plain concrete. Moreover, the most important beneficial effect of fibres on the strength of composites concerns the residual tensile strength. Indeed, fibres bridge tensile stress across matrix cracks, providing considerable residual tensile strength for the composite, which is also present in the presence of significant values of tensile strain. As a result, the width of cracks is reduced by fibre action, increasing the shear capacity of the beam. In the proposed model for the evaluation of the shear strength of SFRC beams with stirrups, the assumption of concrete able to resist compression stress only is removed, and a tensile stress field is introduced. This distribution of tensile stress is assumed to be uniform along the web section and orthogonal to the compression stress field; thus its inclination angle with respect to the longitudinal axis of the beam is equal to $\theta + \pi/2$.

2.2 Lower bound solution for fibrous RC beams

As briefly described in the previous section, the proposed model includes in a clear and easy way the contributions to shear strength due to each strength mechanism. In the following sections, each contribution is defined by an appropriate analytical rule, founded on the plastic theory, aiming to obtain analytical laws for stress distribution at the ultimate limit state (ULS) as a function of the mechanical and geometrical characteristics of the beam, and to provide the beam shear strength by means of the lower bound solution.

To this purpose, the plastic theory provides robust tools, known as theorems of limit analysis. In a previous work (Mancini *et al.* 1996), the analogous plastic model for plain concrete was reported. It was formulated adopting dimensional terms to represent all the features concerning the physical problem. As a consequence, the original analytical equations are lengthy. However, the work of Mancini *et al.* (1996) can be used as a benchmark, because the model proposed in this work is a natural extension of that model to fibrous concrete beams.

Here, to obtain as general a formulation as possible, all geometrical and mechanical quantities involved in the physical problem are expressed in terms of non-dimensional parameters (Fig. 1). The non-dimensional shear and bending moment are defined as $\tau = V/b_w f_{cd2} z$ and $\mu = M/b_w f_{cd2} z^2$ respectively, where b_w is the width of the beam web, z is the lever arm, and f_{cd2} the effective compression strength of the material. Referring to Fig. 1, the variation laws of shear and bending moment, along the longitudinal axes of the structural element, may be expressed as functions of the non-dimensional abscissa $\xi = x/z$ as follows

$$\tau(\xi) = \tau^* - \int_0^{\xi} p(\zeta) d\zeta \tag{1}$$

$$\mu(\xi) = \mu^* + \tau^* \xi - \int_0^{\xi} p(\zeta)(\xi - \zeta) d\zeta \tag{2}$$

τ^* and μ^* being the values of shear and bending moment corresponding to the abscissa $\xi = 0$ respectively; and $p(\xi) = q(\xi)/b_w f_{cd2}$ the law of the non-dimensional distributed load acting on the beam.

Now the shear strength along the beam will be evaluated as a function of the geometrical and

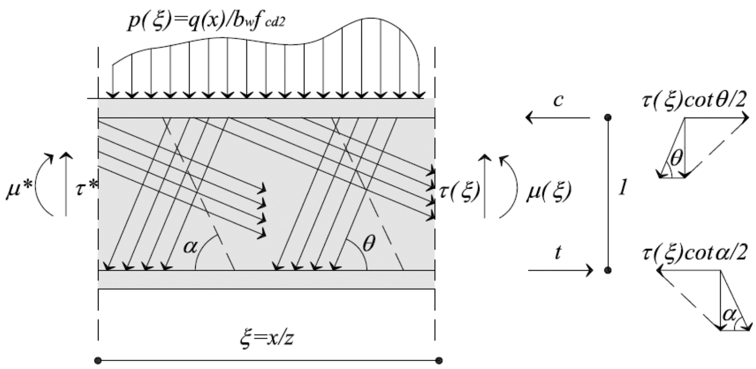

Fig. 1 Shear stress field model

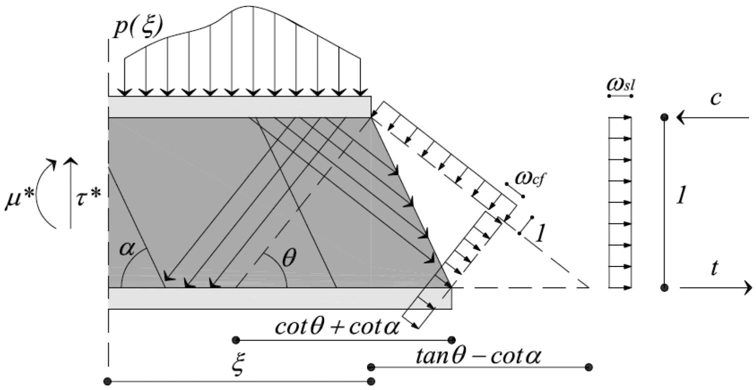

Fig. 2 Cut-section parallel to tensile stress field of stirrups

mechanical characteristics of structural element, and of the compression stress field angle (θ). Thus, a segment of beam ranging from the abscissa $\xi = 0$ and delimited by a cut-section parallel to the tensile stress field of the stirrups (Fig. 2) is considered. The external load $p(\xi)$ causes shear, τ, and bending moment, μ, which are resisted by the inclined compression and tensile stress fields on the concrete web and axial forces on the top ($c = C/b_w z f_{cd2}$) and bottom ($t = T/b_w z f_{cd2}$) chord. The plastic admissible condition at ULS along the transversal direction of the beam at the abscissa ξ reads as follow

$$\tau(\xi) \leq (\cot\theta + \cot\alpha)\sin^2\theta + \omega_{cf}(\tan\theta - \cot\alpha)\cos^2\theta \tag{3}$$

where $\omega_{cf} = f_{ctf}/f_{cd2}$ is the ratio between the residual tensile strength of the fibre reinforced concrete (f_{ctf}), influenced by the presence of fibres in the mixture, and the effective compressive strength (f_{cd2}) at the ULS, $(\cot\theta + \cot\alpha)\sin\theta$ and $(\tan\theta - \cot\alpha)\cos\theta$ are the resultant of the dimensionless compressive and tensile stress fields on the horizontal axis acting on the cut section at the ULS, and the terms on the right-hand side in Eq. (3) are their projections on the vertical axis.

In order to satisfy the plastic limit condition, the shear force $\tau(\xi)$ has to be less than or equal to the plastic shear strength expressed by the whole right-hand side in Eq. (3). Rearranging Eq. (3), the shear strength due to web concrete can be expressed as a function of $\cot(\theta)$

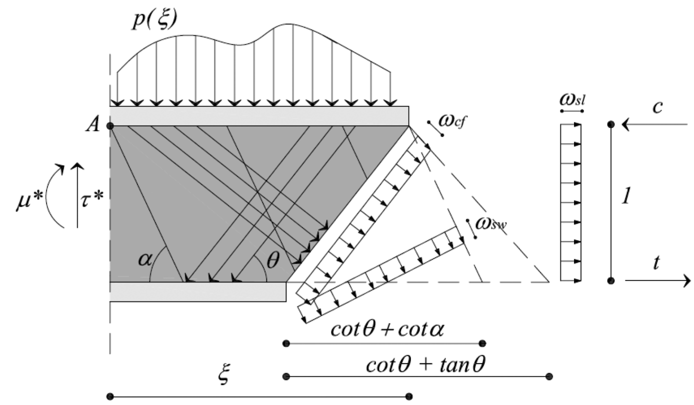

Fig. 3 Cut-section parallel to compressive stress field of web concrete

$$\tau(\xi) \le \frac{\cot\theta + \cot\alpha}{1+\cot^2\theta} + \omega_{cf}\cot\theta\frac{1-\cot\theta\cot\alpha}{1+\cot^2\theta} \tag{4}$$

Likewise, in Fig. 3 there is depicted a segment of beam ranging from the abscissa $\xi = 0$ and delimited by a cut section parallel to the compressive stress field of web concrete induced by the external load $p(\xi)$. In this case the static vertical equilibrium condition returns the limit shear strength that the stirrups are able to withstand

$$\tau(\xi) \le \omega_{sw}(\cot\theta + \cot\alpha)\sin\alpha + \omega_{cf}\cot\theta \tag{5}$$

$\omega_{sw} = (A_{sw}f_{ywd})/(b_w s f_{cd2})$ being the mechanical ratio of stirrups having horizontal space s, section area A_{sw} and yield strength of steel f_{ywd}.

Eqs. (4) and (5) provide the limit values of shear strengths at the ULS corresponding to the collapse of web concrete or stirrups respectively, as a function of the mechanical and geometrical characteristics of the structural element, taking into account the presence of fibres in the mixture by means of simple additional terms depending on ω_{cf}, the angle of compression stress field (θ), and the inclination of the stirrups (α). The top (c) and bottom (t) chords are assumed to resist along their axial direction only. Considering the beam segment in Fig. 3, the rotating equilibrium condition evaluated with respect to the top chord (point A) in conjunction with the plastic admissible condition may be written as follow

$$\begin{gathered}[\mu^* + \int_0^{\xi} p(\zeta)\zeta d\zeta] \le -(\xi - \cot\theta/2)[\omega_{sw}(\cot\theta + \cot\alpha)\sin\alpha + \omega_{cf}\cot\theta] + \\ \frac{1}{2}[\omega_{sw}(\cot\theta + \cot\alpha)\cos\alpha + \omega_{cf}] + \omega_{sl}/2 + t\end{gathered} \tag{6}$$

where $\omega_{sl} = (A_{sl}f_{yld})/(b_w z f_{cd2})$ is the mechanical ratio of skin reinforcement A_{sl} with a yield strength f_{yld}, and $t = T/(b_w z f_{cd2}) = (A_{slb}f_{yd})/(b_w z f_{cd2}) = \omega_{slb}$ the mechanical ratio of bottom longitudinal reinforcement, i.e. the non-dimensional bottom chord strength. With the help of Eqs. (1), (2) and (5), Eq. (6) can be rearranged in the following simpler form

$$\mu(\xi) + \frac{\tau(\xi)}{2}(\cot\theta - \cot\alpha) \le \omega_{slb} + \frac{\omega_{sl}}{2} + \frac{\omega_{cf}}{2}(1 - \cot\alpha\cot\theta) \tag{7}$$

Eq. (7) provides the interaction of bending moment and shear along the abscissa ξ; their limit values are related to the strength of the bottom (ω_{slb}) and surface (ω_{sl}) reinforcement, and the composite tensile strength due to the fibre effect. The influence of external shear force, and the resisting contribution of skin reinforcement (ω_{sl}) and fibre reinforced concrete (ω_{cf}) affect in the same manner the action and the strength of the tensile chords (i.e. the terms on the left and right hand side in Eq. (7) are affected by the coefficient ½) as a consequence of the assumption of a uniform stress field and the model for compression and tensile zones that have been idealized as a stringers carrying concentrated forces (Nielsen 1999).

The bending moment law $\mu(\xi)$ can be expressed as a function of the shear $\tau(\xi)$, mainly depending on the load type and constraint conditions. Thus Eq. (7) can be interpreted as an expression for the shear strength limit. In addition, the formulation makes it possible to evaluate the flexural strength where the shear vanishes and the bending moment is the maximum.

The right-hand side of Eqs. (4), (5) and (7) provides the strength of structural members associate with different collapse mechanisms: crushing of web concrete, yielding of stirrups, or failure at chord level. The limit shear strength is attained in the beam critical section (ξ_0) only, where one or more of Eqs. (4), (5) or (7) is/are satisfied as equality and shear or flexural failure take effect.

Fibres have a beneficial effect in improving the shear strength of all mechanisms, and the proposed model takes this into account by means of simple additive terms, which are essentially a function of the mechanical ratio of the residual tensile strength of the composite (ω_{cf}), the angle of the compression stress field (θ), and the inclination of the stirrups (α).

3. Constitutive behaviour of material

The proposed theoretical model is based on the plastic theory, and one of its fundamental assumptions concerns the constitutive behaviour of material. It is assumed to be always perfectly plastic. However, the effective material strength, coinciding with the plateau value that is typical of a perfectly plastic constitutive curve, is not estimable with a standard uniaxial stress test, and the goodness of the plastic models is strongly influenced by the assumed effective strength values of the material. At failure, the cracked concrete in compression is simultaneously subjected to compression and orthogonal tensile strains. Therefore it exhibits reduced strength compared to uncracked concrete in uniaxially compression. This physical behaviour, namely compression softening (Vecchio and Collins 1986, Nielsen 1999), can be recognized in the plastic theory by the effectiveness factor (ν), which reduces the compressive cylinder concrete strength to the effective value. In addition, as highlighted by Zhang (1997) and Vecchio (2000) for plain concrete, shear slips along cracks further degrade the effective compression strength of material. By contrast, fibres help to contain the width of cracks and shear slips, preserving the strength of the composite (Spinella *et al.* 2010).

3.1 Effective compressive strength

Eurocode 2 (2004) adequately takes into account the physical phenomena described above, adopting the following effectiveness factor (ν) for the concrete in compression of members requiring shear reinforcement (EC2 2004)

$$\nu = 0.6 \quad \text{for} \quad f_{ck} \leq 60 \text{ MPa} \tag{8}$$

$$\nu = 0.9 - f_{ck}/200 \quad \text{for} \quad f_{ck} \geq 60 \text{ MPa} \tag{9}$$

where $f_{ck} = f_{cm} - 8$ (in MPa) is the characteristic cylinder strength.

Few expressions have been proposed to estimate the value of the effectiveness factor in compression of fibrous concrete (Spinella *et al.* 2010, Voo *et al.* 2006). Fibrous concrete shows uniaxial compression strength similar to plain concrete, but less softening in the post-peak branch is always observed when a small quantity of steel fibres is placed in the mixture (Nataraja *et al.* 1999). Thus for fibrous concrete a higher value of the effectiveness factor in compression than for plain concrete is reasonably expected. A recalibration of the effectiveness factor for fibrous concrete in compression would be needed, but it would require a large testing campaign. However, for the proposed model the effectiveness factor ν for plain concrete, as defined in Eqs. (8) and (9), is also adopted for SFRC, providing a lightly conservative evaluation of the effective compressive concrete strength.

3.2 Effective tensile strength

As previously mentioned, the constitutive behaviour of fibrous concrete under direct tensile force is different than that of plain concrete, specially in the cracking phase. In fact, the residual tensile strength of the material is negligible for plain concrete; by contrast, it takes on a significant value for fibrous concrete, and it is practically constant after the first cracking of the matrix, also for high tensile strain values (Vandewalle 2002). Thus it can be modelled by a plastic constitutive behaviour. Fibres provide the bridging of stress across cracks due to the pullout process, which provides an improvement of the toughness of the composite and an increasing in fracture energy before collapse, depending on the bond-slip behaviour at the interface between fibre and matrix.

Most of the analytical laws proposed in literature to model the constitutive behaviour of fibrous concrete under direct tensile stress are derived from the pull-out mechanism of a single fibre embedded in the matrix. The pull-out curve is usually evaluated by means of experimental tests (Banthia and Trottier 1994), but in some interesting models theoretical pull-out curves are derived (Foster *et al.* 2006, Lim *et al.* 1987). Lim *et al.* (1987) derived a theoretical pull-out curve for a single fibre embedded in the matrix, and then a generalization to the case of more random fibres dispersed in the mixture was performed. The model was validated by several experimental tests carried out by the authors on different SFRC specimens subjected to direct tension.

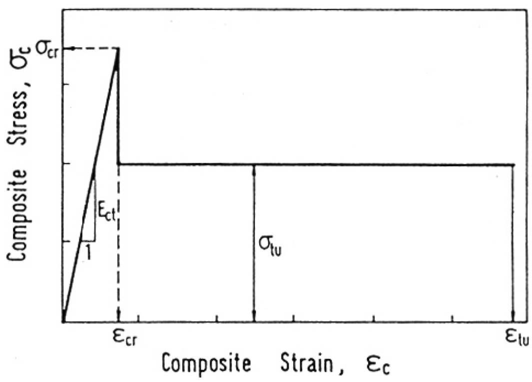

Fig. 4 Analytical constitutive tensile law proposed by Lim *et al.* (1987)

The shape of σ-ε constitutive curve proposed by Lim *et al.* (1987) for fibrous concrete under tensile stress is shown in Fig. 4. The first branch is linear up to the tension value σ_{cr} corresponding to the first crack in the matrix; then the post-peak phase is modelled with a plateau representing the constant value of residual tensile strength (σ_{tu}), which is always smaller than the cracking stress value (σ_{cr}). This shape of the post-peak branch makes this constitutive model suitable for the application of the plastic theory. Thus in the proposed model the prediction of the strength of SFRC beams with stirrups is achieved by using the theoretical constitutive tensile law proposed by Lim *et al.* (1987).

For the residual tensile strength value (σ_{tu}), the following analytical expression is suggested (Lim *et al.* 1987)

$$\sigma_{tu} = 2\,\eta_0\,\eta_l V_f (l_f/d_f)\,\tau_f \tag{10}$$

where V_f is the volumetric percentage of fibres; l_f/d_f is the aspect ratio between the length (l_f) and the diameter of fibre (d_f), respectively; τ_f is the average bond stress between matrix and fibre; $\eta_0 = 0.405$ is the orientation factor; and η_l is the length efficiency factor of the fibre, which depends on the critical length $l_c = (\sigma_{fu}\, d_f)/(2\tau_f)$: if l_f is less than or equal to l_c then $\eta_l = 0.5$, else $\eta_l = (1 - l_c/2l_f)$ (Lim *et al.* 1987). In the proposed model the residual tensile strength of fibrous concrete (f_{ctf}) is assumed to be equal to the residual tensile strength (σ_{tu}), while the tensile strength of the matrix (f_{ct}) matches the tension at the first crack of the matrix (σ_{cr}).

3.2.1 The fibre factor F_τ

As in most of the analytical constitutive tensile laws for SFRC proposed in the literature, in Lim *et al.*'s model knowledge of the average bond stress between matrix and steel fibre (τ_f) is needed in order to calculate the residual tensile strength (10). Many experimental and analytical works have been published in the literature about this complex issue, suggesting several values (Swamy and Bahia 1985, Narayanan and Darwish 1987) or calculation methods (Valle and Büyüköztürk 1993) for τ_f. Voo and Foster (2003), analyzing a large database of experimental results, observed that the average bond stress is strictly related to the fibre shape and the tensile strength of the matrix (f_{ct}), as previously stated by Valle and Büyüköztürk (1993). On the basis of this large work, Voo and Foster (2003) suggested simple analytical relationships between the tensile strength of the matrix (f_{ct}) and the average bond stress (τ_f), as a function of the fibre type (hooked or straight). Those relationships are rearranged here and introduced in the original expression of the residual strength (9), which is rewritten as follows

$$f_{ctf} = 2\,\eta_0\,\eta_l F_\tau f_{ct} \tag{11}$$

where $F_\tau = V_f (l_f/d_f)\beta_\tau$ is the fibre factor (Spinella *et al.* 2010), and $\beta_\tau = \tau_f/f_{ct}$ is equal to 2.5 or 1.2 for hooked or straight fibre embedded in concrete matrix, respectively (Voo and Foster 2003). Now, the residual tensile strength for fibrous concrete (11) depends directly on the tensile strength of the matrix (f_{ct}), which is assumed to be equal to $0.45 f_{\mathrm{cm}}^{0.4}$ (in MPa) as suggested by Bentz (2000).

The fibre factor, F_τ, provides a measure of the ability of fibre to confer toughness on the composite and reaches its maximum value, $F_{\tau,\,max} = 1/(2\eta_0\eta_l)$, when the residual tensile strength of the composite matches the tensile strength of the matrix, i.e. the equality $f_{ctf} = f_{ct}$ in Eq. (11) is verified. To investigate the influence of the length efficiency factor, η_l, on the fibre factor and on the residual tensile strength provided by fibres, a sensitivity analysis was carried out and is shown

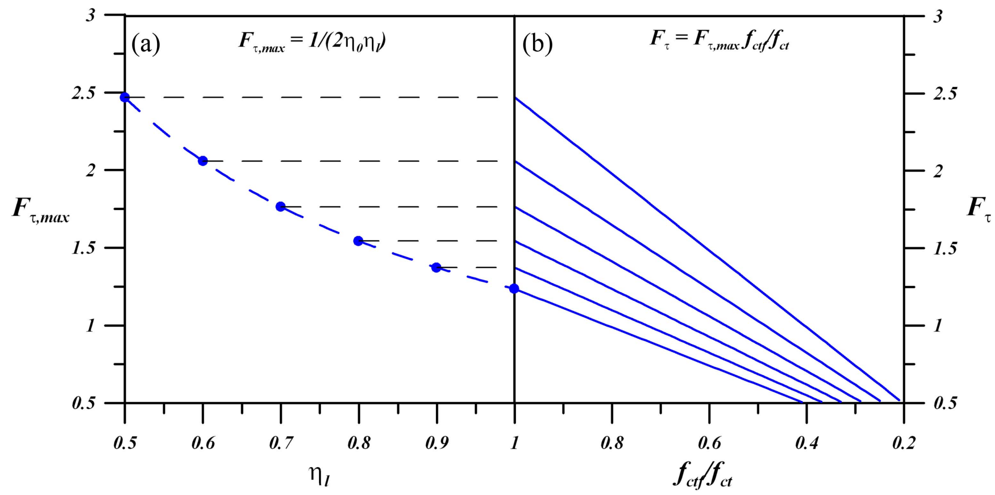

Fig. 5 Non-dimensional residual tensile strength of composite as a function of fibre and concrete properties

in Fig. 5. In Fig. 5(a), $F_{\tau,\,max}$ is plotted for various values of the length efficiency factor of the fibre, η_l, which depends on the geometrical and mechanical characteristics of fibres and concrete. Increasing the value of η_l (i.e. improving the bond strength between fibre and matrix), the maximum value of the fibre factor decreases. In Fig. 5(b) the curves of the fibre factor, F_τ, versus the $f_{ctf}/f_{ct} = F_\tau/F_{\tau,\max}$ ratio are depicted. Once the geometrical (l_f and d_f) and mechanical properties (σ_{fu}) of fibre and concrete (f_{ct}) are known, the length efficiency factor (η_l) and the aspect ratio (l_f/d_f) are defined. The length efficiency factor allows one to evaluate $F_{\tau,\,max}$ in Fig. 5(a) and the corresponding line in Fig. 5(b). Varying the volumetric percentage of fibre (V_f), the fibre factor F_τ is easily evaluated ($F_\tau = V_f \beta_\tau l_f/d_f$) and the f_{ctf}/f_{ct} ratio can be calculated.

4. Application of the proposed model

The proposed model is now completely defined by Eqs. (4), (5) and (7), which represent the strength limits of the beam for different collapse mechanisms. Furthermore, all mechanical and constitutive parameters that appear in the equations are defined in an explicit way. The presence of fibres is taken into account by the parameter w_{cf}. If this value is equal to zero (no fibres in the mixture, so that no residual tensile strength is considered) the formulation matches the model proposed for plain concrete beams (Mancini *et al.* 1996). The analytical formulation at the basis of the proposed model to evaluate the shear strength of SFRC beam with stirrups is simple. Further, it provides interesting information on the collapse mode of beams and the residual capacity of each shear strength mechanism.

4.1 Shear strength domain

The variation laws of strength mechanism capacity relating to crushing of web concrete (4), yielding of stirrups (5), and failure at chord level (7) were written to be plotted in a τ-$Cot(\theta)$ plane. They define a shear strength domain in the range $1 \le Cot(\theta) \le 2.5$, as suggested by EC2 (2004), and depending on the mechanical and geometrical characteristics of the structural element. In order to plot the curve of the shear strength provided by Eq. (7) the choice of the critical cross-section (ξ_0), where the failure is expected, is needed. In Fig. 6 a beam scheme for a four-point load test and the corresponding variation in shear and bending moment are shown. In the region under the load point the compressive and tensile stress fields assumed at the basis of the proposed model are not developed. Then the width of this discontinuity region, denoted with letter *D* in Fig. 6, is assumed to be equal to the height (*h*) of the beam, as suggested by EC2 (2004). As a consequence of these observations, the critical shear section is chosen in the *B* region in the section where shear and bending moment reach their maximum value: $\xi_0 = (a\text{-}h)/z$. Moreover, for the beam-scheme considered, the shear-bending moment interaction along the shear span of the beam can be expressed as $\mu(\xi) = \tau(\xi)\xi$. Thus Eq. (7) can be rearranged obtaining the analytical expression of the shear strength

$$\tau(\xi) \leq \frac{2\omega_{slb} + \omega_{sl} + \omega_{cf}(1 - \cot\alpha\cot\theta)}{2\xi + \cot\theta - \cot\alpha} \tag{12}$$

By contrast, failure due to bending moment is likely in the mid-span region of beam; then the flexural strength for the four-point load scheme is evaluated setting equal to zero the shear value in Eq. (7)

$$\mu(\xi) \leq \omega_{slb} + \frac{\omega_{sl}}{2} + \frac{\omega_{cf}}{2}(1 - \cot\alpha\cot\theta) \tag{13}$$

In Figs. 7(a) and 7(b) graphical representations of the equations proposed are reported, assuming that stirrups are placed with an inclination angle on the horizontal axis of 45° or 90° (as is usual in common practice), respectively. The dashed line curves define the shear domain for a plain concrete beam ($\omega_{cf} = 0$), while solid line curves define the analogous domain for fibrous concrete ($\omega_{cf} = 0.05$). In the same Figs, the mechanical percentage of stirrups (ω_{sw}) is kept constant at the value of 0.05, while the non-dimensional abscissa of the shear critical section (ξ_0) is assumed to be equal to 1.00.

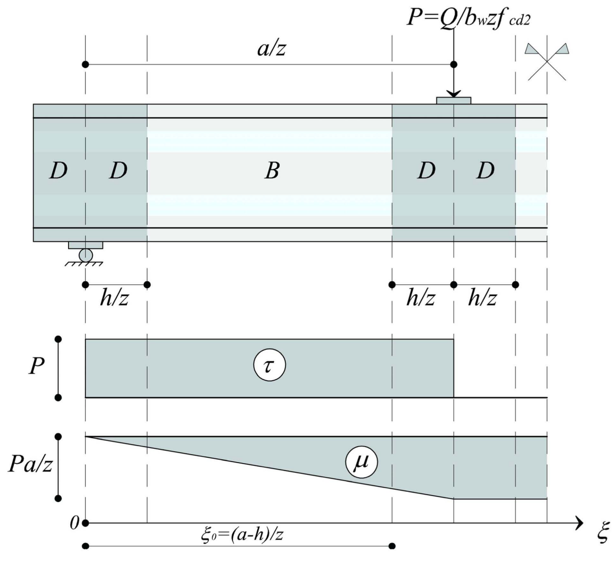

Fig. 6 Bernoulli (*B*) and Discontinuity (*D*) regions

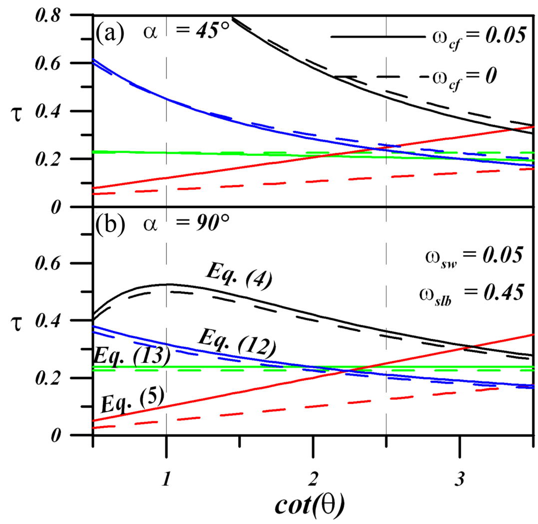

Fig. 7 Shear strength domain for stirrups inclined (a) 45° and (b) 90°

The area between two analogous curves plotted for plain and fibrous concrete, respectively, gives a measure of the strength increase due to fibres related to each collapse mechanism. The beneficial effect of fibres is clear, especially for a beam with stirrups inclined 90° with respect to its longitudinal axis and for the collapse mechanism related to yielding of stirrups (5). This highlights and confirms the ability of fibres to help stirrups as shear reinforcement. The shear strength mechanisms related to concrete crushing (4) and chord failure (12) show a small and constant increase in τ due to the fibre for every value of $Cot(\theta)$. By contrast, the increment due to the fibre in beams where shear strength is ruled by stirrup failure (5) is greater for high values of $Cot(\theta)$ (i. e. high slope of compression stress field). Similar considerations may be made for a beam with stirrups inclined 45° with respect to its longitudinal axis. However, the enhancement of shear strength due to the use of fibres is less than its counterpart in beams with stirrups inclined 90°, but the trend for each curve is the same.

The ordinate of the intersection points between curves (4), (5) and (12) inside the admissible range of $Cot(\theta)$ provides the shear strength values allowed for the structural element: thus the effective shear strength of the beam is the minimum value among them. It has to be compared with the flexural strength evaluated by Eq. (13) to identify the effective failure mechanism of the beam and the corresponding collapse load.

The proposed model provides the collapse mode. For example, when the intersection point related to the shear strength of the beam is obtained as the intersection of curves (5) and (12), as in Fig. 7b, ductile failure is caused by the simultaneous yielding of stirrups and equivalent skin reinforcement before the flexural strength is reached. As a function of the parameters involved, the failure can be due to crushing of concrete and yielding of stirrups or surface reinforcement. If the beam is heavily reinforced in shear, the crisis is due to crushing of concrete with stress level of steel reinforcement in both directions (transversal and surface rebar) under the yield limit, and the failure of the beam is brittle.

5. Validation of proposed model

With the aim of validating the proposed model for the evaluation of the shear strength of FRC beams with stirrups, a wide database of around fifty experimental test results (Swamy and Bahia 1985, Furlan and Bento de Hanai 1997, Campione and Mindess 1999, Dupont and Vandewalle 2003, Campione *et al.* 2004, Kearlsley and Mostert 2004) was collected from the literature. The database used for validation of the proposed model is representative of large variations in all parameters influencing the response of SFRC beams with stirrups under shear actions. In fact, variation in the fibre aspect ratio (l_f/d_f) in the range of 40 to 133 was considered, together with a volume percentage of fibres (V_f) between 0.38% and 2.00%, as in the most common practical applications of steel fibres in the construction field, and the geometrical percentage of stirrups (p_w) in the range of 0.09% to 1.55%.

5.1 Corroboration and comparison with other models

In Fig. 8(a) the comparison between the experimental and analytical results for fibrous concrete beams provided by the proposed model is shown. The analytical model for the estimation of the shear strength appears to be very effective in predicting the load-bearing capacity of SFRC beams, and the mean and Coefficient of Variation (COV) values are 1.06 and 0.23, respectively. Most of the known formulations for the evaluation of the shear strength of SFRC beams with stirrups are obtained by numerical regression on a test database, and consequently their range of use is often limited to the range of variation of the parameters investigated during the tests, so that they are not able to capture the physical mechanism of failure. However, some power equations are based on a macroscopic description of phenomena involved in the strength mechanism, and then the numerical coefficients are calibrated by statistical analyses; they may be classified as semi-empirical models, such as those recently proposed by Campione *et al.* (2006) (CLP06) and Thomas and Ramaswamy (2006) (TR06). These two models were considered in order to carry out a further comparison on the effectiveness of the shear strength prediction.

Campione *et al.* (2006) extend a formulation given in the literature for plain concrete beams based on an evaluation of the strength contribution of beam and arch actions in the case of fibrous concrete beams. The CLP06 model is worthy of note because it introduces a parameter able to measure the shear stress shared between stirrups and steel fibres. Thomas and Ramaswamy (2006) proposed a model derived from the regression of 518 test data for the prediction of the shear strength of reinforced and prestressed concrete beams with different cross-section shapes, with and without shear reinforcement such as stirrups and/or steel fibres. The TR06 model is able to take into account the presence of fibres over the partial or full depth of the beam. Despite its semi-empirical nature, the TR06 model allows one to evaluate the shear strength of structural elements with several geometrical and mechanical characteristics.

The results shown in Figs. 8(b) and 8(c) prove that the models considered, despite their semi-empirical nature, provide good estimations of the shear capacity of SFRC beams with stirrups. However, the mean and COV obtained with the considered semi-empirical models are still higher than the corresponding values obtained with the proposed analytical model. The theoretical basis of formulation proposed here is able to reproduce each shear strength mechanism and its interactions, providing realistic values of shear capacity of the beam and useful information about the collapse mode of the structural element.

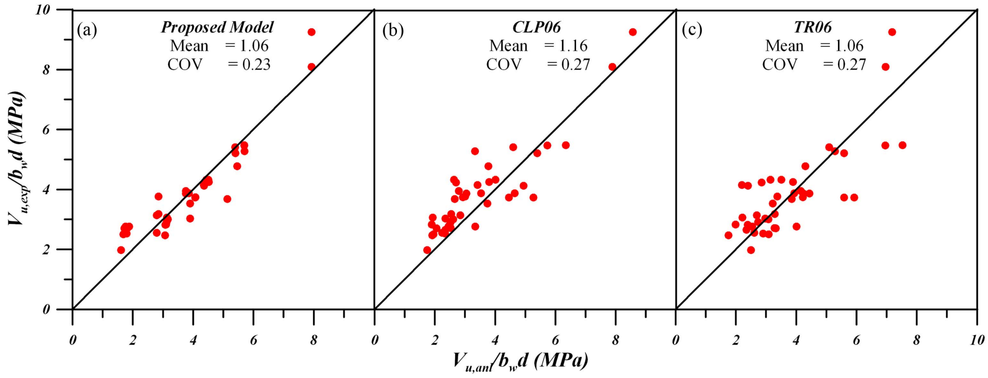

Fig. 8 Comparison between experimental and analytical results using (a) the proposed model, (b) the CLP06 model, and (c) the TR06 model

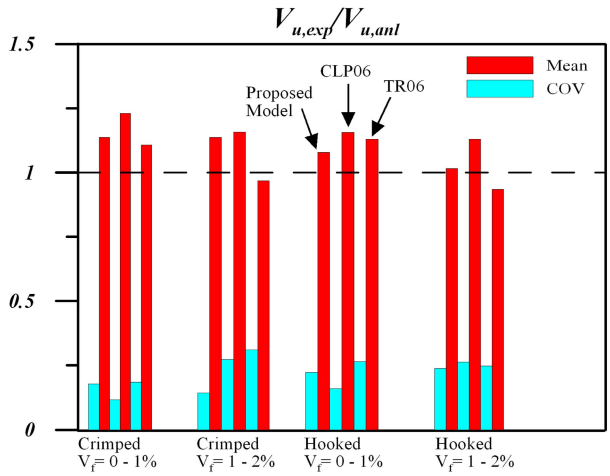

Fig. 9 Comparison between experimental and analytical results differentiating types and volume percentages of fibres

Fig. 9 sums up the comparison with variation in the tests on different types (crimped and hooked) and with different volumes (0-1% and 1-2%) of fibres, showing that the proposed model provided a good performance in all cases considered.

5.2 Numerical analysis of failure mechanism

Fibre reinforcement enhances shear resistance by transferring tensile stress across diagonal cracks and reducing diagonal crack spacing and width, which increases aggregate interlock. Consequently, the inclination of the compressive stress field with respect to the horizontal axis (θ) is smaller for fibrous concrete beams than for analogous plain concrete beams, proving to be closer to 45 degrees, as can be recognized by comparing the curves in Fig. 7 for the predicted value of $Cot(\theta)$ at collapse for fibrous and plain concrete.

With the aim of investigating the capacity of the proposed model to reproduce the ability of fibres

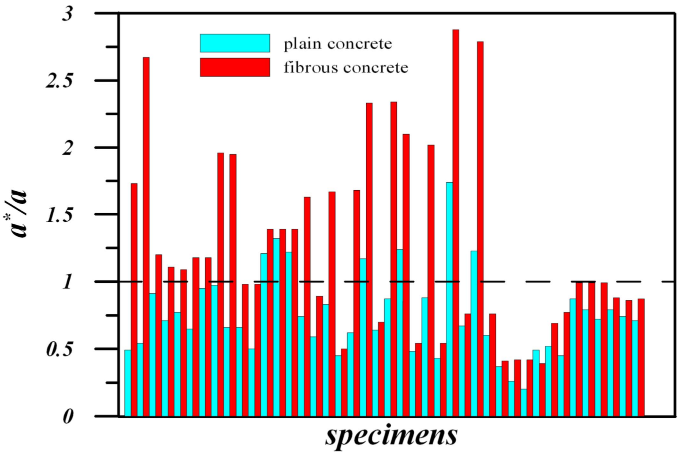

Fig. 10 Comparison between numerical shear span ratio (a^*/a) of plain and fibrous concrete beam

to change the failure mode of beam from brittle for shear to ductile for flexure, a numerical analysis was carried out. All specimens taken into account for the validation of the proposed model are now considered with and without fibre in the concrete mixture. On the basis of the experimental results available, first the capacity of the analytical model to reproduce the behaviour of regular concrete beams was checked. The analysis carried out on the database of available specimens (18 beams) provided a mean and COV of predicted shear strength values of 1.02 and 0.22, respectively, confirming the goodness of the analytical model.

Secondly, a critical shear span (a^*) for which the flexural and shear strength have the same value is calculated by an iterative procedure, for both plain and fibrous concrete specimens. To calculate the critical shear span a^*, the position of the vertical load was moved along the abscissa ξ, until the difference between flexural and shear strength was less than a prefixed error. Denoting with a the actual shear span used during the experimental test, the ratio between the critical and actual shear span (a^*/a) was evaluated; values of the (a^*/a) ratio greater than one indicated a predicted brittle collapse of the beam, and values smaller than one a predicted ductile failure mode.

In Fig. 10, two histograms are plotted showing the shear span ratio obtained by analysis of plain and fibrous concrete beams, respectively. Observing the results it can be recognized that the proposed model is able to predict the change in failure mode due to the presence of fibres. The value of the ratio a^*/a is more than one in most of the regular concrete beams considered, proving that they are critical in shear, as expected. Considering the fibre reinforcement for each corresponding beam, the shear span ratio decreased in all cases; hence the analytical model was able to reproduce the beneficial effect of fibres in terms of shear strength. In some cases, the ratio a^*/a reached values of less than one, and thus the analytical model predicted ductile collapse.

6. Conclusions

A theoretical model for evaluation of shear strength of SFRC beams with stirrups has been proposed. The beneficial contribution of fibres to each shear strength mechanism is mainly taken into account by introducing the tensile stress field in the concrete web. An analytical constitutive tensile law has been adequately modified to evaluate the residual tensile strength of fibrous concrete, as a function of the fibre factor F_{τ} which takes into account the amount and mechanical and geometrical characteristics of fibres.

The validation against a wide database of experimental test results published in the literature and a comparison with two well known semi-empirical models showed the goodness of the proposed formulation, obtaining very small values of mean error and COV. The effectiveness of the model proves the consistency of the physical role of the tensile stress field considered and the efficiency of the residual tensile strength relationship assumed for SFRC.

The graphical interpretation of the proposed equations allows an understanding of the interaction between the shear strength mechanisms involved. It also shows the capacity of fibres to keep the inclination of the compression stress field close to 45°, and consequently to promote ductile failure. The graphical representation of the analytical relationship proposed leads to a simple procedure for the optimal design of shear reinforcement made up of both steel fibres and stirrups.

Finally, the results of numerical analyses carried out to evaluate the shear span (a^*) for which the flexural and shear strength have the same value highlights the ability of the proposed model to predict the change in beam failure mode from brittle to ductile due to fibre reinforcement.

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Nomenclature

a, a^*	Effective and critical shear span
A_{sl}	Skin reinforcement area
A_{sw}	Stirrup reinforcement area
b_w	Width of beam cross-section
c, C	Non-dimensional and dimensional top chord forces
d, h	Effective depth and full height of beam cross-section
d_f, l_f	Diameter and length of fibre
f_{cd2}	Effective compression strength
f_{ck}, f_{cm}	Characteristic and mean cylinder strength
f_{ct}, f_{ctf}	Tensile strength of matrix and residual tensile strength of composite
f_{yld}	Yield strength of steel surface reinforcement
f_{ywd}	Yield strength of steel stirrup
F_t	Fibre factor
l_c	Critical length of fibre
M	Dimensional bending moment
p, q	Non-dimensional and dimensional uniform distributed loads
P, Q	Non-dimensional and dimensional nodal loads
R_{ck}	Characteristic cubic strength of concrete
s	Stirrup spacing
t, T	Non-dimensional and dimensional bottom chord forces
V	Shear
V_f	Volumetric percentage of fibres
x	Abscissa along longitudinal axis of beam
z	Lever arm
α	Inclination of stirrups with respect to horizontal axis of beam
α_c	Coefficient for prestressed structures
β_τ	Fibre bond factor
η_0, η_l	Orientation and length efficiency factors of fibre
μ	Non-dimensional bending moment
μ^*	Non-dimensional bending moment at specific value of ξ
ν	Effectiveness factor
θ	Inclination of compressive stress field with respect to the horizontal axis of the beam
σ_{cr}, σ_{tu}	First crack and residual tensile strength of material (Lim *et al.* 1987 model)
τ	Non-dimensional shear
τ^*	Non-dimensional shear at specific value of ξ
τ_f	Average bond stress between matrix and fibre
ω_{cf}	f_{ctf}/f_{cd2} = Mechanical ratio of residual tensile strength of composite
ω_{sl}	$(A_{sl}f_{yld})/(b_w z f_{cd2})$ = Mechanical ratio of skin reinforcement
ω_{slb}	Mechanical ratio of bottom longitudinal reinforcement
ω_{sw}	$(A_{sw}f_{ywd})/(b_w s f_{cd2})$ = Mechanical ratio of transversal reinforcement
ξ	Non-dimensional abscissa
ξ_0	Non-dimensional critical section of abscissa