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Modelling inelastic hinges using CDM for nonlinear analysis of reinforced concrete frame structures

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Abstract. A new formulation based on lumped plasticity and inelastic hinges is presented in this paper for nonlinear analysis of Reinforced Concrete (RC) frame structures. Inelastic hinge behaviour is described using the principles of Continuum Damage Mechanics (CDM). Member formulation contains provisions to model stiffness degradation due to cracking of concrete and yielding of reinforcing steel. Depending on its nature, cracking is classified as concentrated or distributed. Concentrated cracking is accounted through a damage variable and its growth is defined based on strain energy principles. Presence of distributed flexural cracks in a member is taken care of by modelling it as non-prismatic. Plasticity theory supported by effective stress concept of CDM is applied to describe the post-yield response. Nonlinear quasi-static analysis is carried out on a RC column and a wide two-storey RC frame to verify the formulation. The column is subjected to constant axial load and monotonic lateral load while the frame is subjected to only lateral load. Computed results are compared with those due to experiments or other numerical methods to validate the performance of the formulation and also to highlight the contribution of distributed cracking on global response.

Keywords: reinforced concrete frame; plastic hinges; continuum damage mechanics; plasticity theory; nonlinear response.

1. Introduction

Nonlinear response of a Reinforced Concrete (RC) frame structure is governed by cracking and yielding of its members. Analysis for nonlinear response of a frame structure is possible based on 3D solid continuum approach (Bazant 2002) or lumped plasticity approach (Mwafy and Elnashai 2001). Among these, models based on the latter approach provides an effective means for nonlinear analysis of a large RC frame structure (Riva and Cohn 1990, Kunnath *et al.* 1990). It exploits local presence of the cracking and yielding zones which induces softening behaviour of the structure. A typical model of this approach consists of inelastic hinges embedded within a predominantly elastic region. As general solution to such problems, Bolzon and Corigliano (1997) have proposed a

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unified discrete formulation by combining perfectly-plastic and softening hinges with suitable damaging interface. The formulation is applied to solve different class of problems with computational background. On that basis, Ehrlich and Armero (2005) have analysed beams and frames for their nonlinear response using softening plastic hinges in a finite element model. According to this, geometric definition of a frame member is made with an elastic element appended at either ends by two inelastic hinges. With such simple geometric definition of the structure, the results of the analysis depend primarily on the deformation characteristics defined for the inelastic hinges. Both material (Tangramvong and Tin-Loi 2008, Park and Ang 1985) and geometry (Tangaramvong and Tin-Loi 2009) nonlinearity can be accommodated while defining the properties of inelastic hinges.

Deformation behaviour of the hinges is controlled through proper definition of moment-curvature relation of the section at which these are expected to form. This has to be superimposed with a compatible degradation model to realise degrading response of the structure. Mainly models based on degrading the stiffness of members are used due to their simplicity in modelling and the possibility to directly correlate with the physical behaviour of the structure. Three parameter degradation model proposed by Park and Ang (1985) and adopted in IDARC-2D (2004) software is an example for this. The parameters are chosen based on evidence of degradation in number of experimental results (Kunnath *et al.* 1991). However, need to decide suitable values for the parameters limit the use of such models in the nonlinear analysis of structures.

By being devoid of such parameters, a Continuum Damage Mechanics (CDM) based degradation model associated with inelastic hinges presents generality in modelling and analysis of a RC frame structure (Florez-Lopez 1998). It operates using a well-defined damage variable to quantify the reduction in stiffness (alternatively the additional flexibility) w.r.t. excessive response. Subsequently, numerical value of the damage variable is linked to member response through a damage function. CDM principles can be used to describe the inelastic behaviour of hinges in a simplified approach for nonlinear analysis of RC frame structures (Cipollina *et al.* 1995). As shown later in present study, however, this model is not able to show finite reduction in member stiffness which happens just after the moment demand on the member exceeds the cracking capacity. Existence of cracks distributed over a short segment near the ends of a member is ignored in the definition of the elastic component of that model.

In the present context, formation of flexural cracks that results in segregation and discontinuity of concrete is considered as damage. With the objective of developing a model, both concentrated and distributed cracking (damage) in members are considered as consequences of the member response which is induced by load. Gradual reduction in the member stiffness is considered as the effect of cracking (Fig. 1).

With this background, a formulation for nonlinear analysis of RC frame structures is proposed in this paper. It depends on CDM for the inelastic hinges and a refined model for the elastic component. Tri-linear moment-curvature relation of a section is taken as the basis to define the

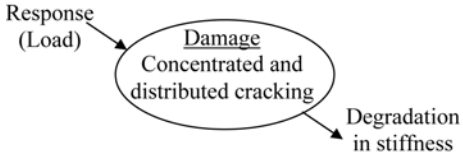

Fig. 1 Damage - its cause and effect

deformation characteristics of a hinge. Degradation is modelled through an isotropic damage variable within the paradigm of CDM. Additional flexibility arising due to hinge behaviour is proposed to be augmented with flexibility of the elastic member. Growth of damage due to intense cracking at ends is introduced through a damage function available in literature. Plastic deformation due to yielding of reinforcement is expressed as a function of the moment in excess of the yielding moment capacity of the section. Elastic member is assumed to be either uncracked or containing distributed cracks. The present model differs from the comparable ones by introducing a refinement to the elastic member definition.

Non-prismatic member definition is used for elastic component of the model. Flexural stiffness defined by the tri-linear moment-curvature relation of critical sections is taken as the basis to evaluate the elastic flexibility of the member. Verification of the formulation is done by evaluating the nonlinear response of a i) RC column for two different axial load levels and ii) a wide two-storey RC frame. Nonlinear variation of base shear with the applied displacement is compared with literature values to recognize the performance of the model. Intermediate results of diagnostic nature about the column response are presented and discussed to highlight the capabilities of the model and the analysis.

2. Continuum damage mechanics

CDM is concerned with modelling the behaviour of solids, that contain distributed cracks, to external loads. CDM proposes to evaluate effective stress (Krajcinovic 1996) ($\bar{\sigma}$) in a damaged material by

$$\bar{\sigma} = \frac{\sigma}{1-D} \tag{1}$$

where σ is uniaxial stress found using undamaged cross section area and D is internal damage variable that numerically represents the damage in a member. Constitutive equations of a damaged material may be derived using effective stress instead of actual stress as per strain equivalence principle of CDM (Krajcinovic 2000, Lemaitre and Desmorat 2005). Effective stress concept and strain equivalence principle have enabled successful application of CDM for solution of practical engineering problems. Damage variable is defined in a model based on CDM to describe the degradation in stiffness and/or reduction in strength of a structural component. The internal variable can either be a scalar to model isotropic damage or a tensor to represent anisotropic damage.

2.1. Formulation

A nonlinear formulation for quasi-static analysis of RC frames is presented in this section. A typical frame structure subjected to external loads is shown in Fig. 2. The frame geometry is defined by a set of 'M' members and 'N' nodes. The geometric configuration and the usual method of applying external loads as work equivalent values at nodes in the analysis produce maximum moment demand at the member ends. Therefore, the frame members are expected to experience initial flexural cracking at member ends. Depending on the moment distribution, flexural cracks may be distributed over a region near the ends of a member.

Only the nonlinearity due to material behaviour is included in the formulation. Geometric

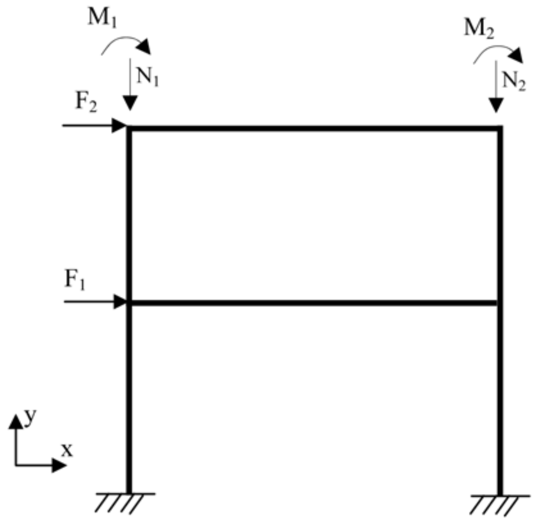

Fig. 2 A simple frame structure subjected to external loads

nonlinearity is proposed to be considered in the analysis through P-Δ effect. A frame member m is uniquely defined by a pair of nodes (i, j) between which it is spanning.

Deformed configuration of a column or a beam (Figs. 3a and 3b) of the member set [0, M] is defined by

$$\{q\}_{col}^{T} = \{u_i,\ \phi_i,\ v_i,\ u_j,\ \phi_j,\ v_j\} \tag{2a}$$

$$\{q\}_{beam}^{T} = \{v_i,\ \phi_i,\ v_j,\ \phi_j\} \tag{2b}$$

where u_i and v_i are the components of displacements of node i along x- and y-axes; ϕ_i is the end rotation of a member connected to node i.

The subscripts 'col' and 'beam', respectively, denote that the vectors are meant for a column and a beam. The displacement vector $\{r\}$ of the frame structure is formed by accumulating the displacement components of all the nodes in the frame structure, i.e.,

$$(r)^T = \{u_1, \phi_1, v_1, \ldots, u_i, \phi_i, v_i, \ldots, u_N, \phi_N, v_N\} \tag{3}$$

where $0 < i < N$

Under dominant lateral load compared to vertical load, a beam member of the frame can be considered to deform in anti-symmetric bending mode (Meyer *et al.* 1998). This consideration is also mentioned to be suitable for columns in low-rise frames. Based on this, undeformed and deformed configurations of a typical column and beam along with external forces are shown in Figs. 3(a) and 3(b). A column is subjected to bending moments, shear and axial forces while a beam is subjected to bending moments and shear forces only. Then equilibrium of a member is established by equating the internal and external forces and moments, as the case may be, corresponding to a column or a beam

$$[S_m]\{Q_m\} = \{M\} \tag{4}$$

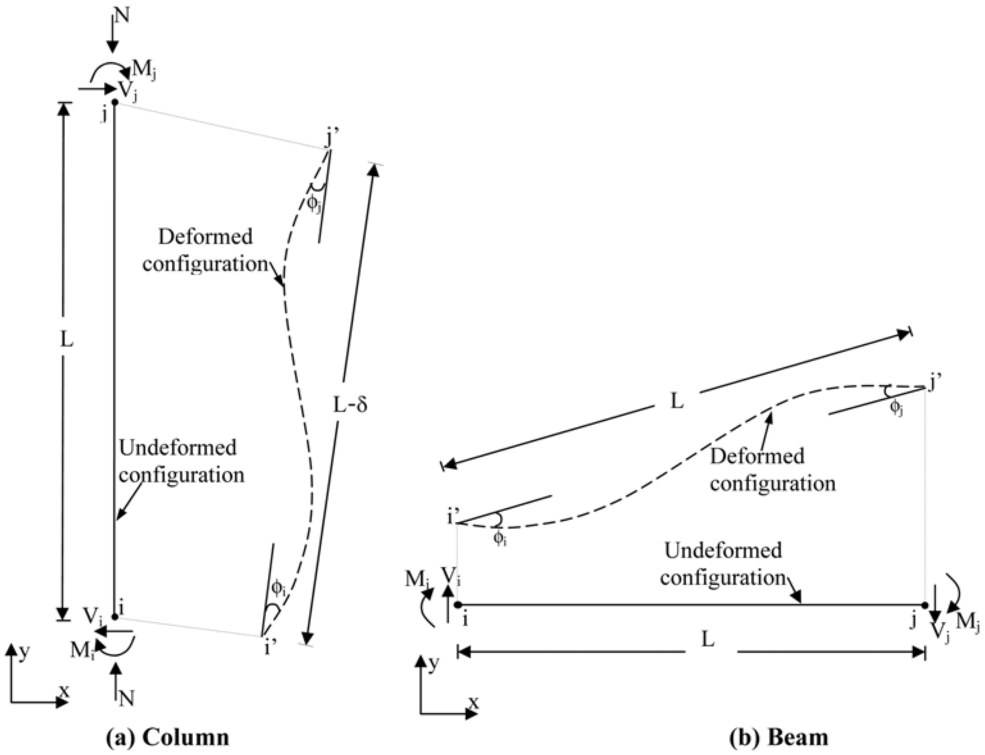

Fig. 3 Generalised forces and corresponding deformations

where $[S_m]$ is the member stiffness matrix; $\{Q_m\}$ is the deformation vector of a member as given below for a column and a beam

$$\{Q\}^T_{\text{col}} = \{\delta, \phi_i, \phi_j\} \tag{5a}$$

$$\{Q\}^T_{\text{beam}} = \{\phi_i, \phi_j\} \tag{5b}$$

where δ is change in length of the column; ϕ_i and ϕ_j are rotations of member ends as shown in Figs. 3(a) and 3(b); $\{M\}$ is vector of forces and moments in a column or beam as given below

$$\{M\}^T_{\text{col}} = \{N,\ M_i,\ M_j\} \tag{5c}$$

$$\{M\}^{\text{T}}_{\text{beam}} = \{M_i,\ M_j\} \tag{5d}$$

where N is the axial force; M_i and M_j are the bending moments at ends i and j.

A transformation matrix is introduced to relate the deformation components of a member with its deformation vector. The relation is expressed for a column and beam as

$$\{Q\}_{\text{col}} = [B]_{\text{col}}\{q\}_{\text{col}} \tag{6a}$$

$$\{Q\}_{\text{beam}} = [B]_{\text{beam}}\{q\}_{\text{beam}} \tag{6b}$$

where $[B]_{\text{col}}$ and $[B]_{\text{beam}}$ are the transformation matrices corresponding to a column and a beam. For orthogonal alignment of columns and beams along y- and x-axes, respectively, as shown in Figs.

3(a) and 3(b), the transformation matrix is

$$[B]_{\text{col}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & -1 \\ \frac{1}{L} & 1 & 0 & -\frac{1}{L} & 0 & 0 \\ \frac{1}{L} & 0 & 0 & -\frac{1}{L} & 1 & 0 \end{bmatrix} \tag{7a}$$

$$[B]_{\text{beam}} = \begin{bmatrix} -\frac{1}{L} & 1 & \frac{1}{L} & 0 \\ -\frac{1}{L} & 0 & \frac{1}{L} & 1 \end{bmatrix} \tag{7b}$$

Substituting Eqs. (6a) and (6b) into Eq. (4) and premultiplying the resulting equation by $[B_m]^{\text{T}}$, it can be written in a unified sense for a column or beam as

$$[B_m]^T[S_m][B_m]\{q\} = [B_m]^T\{M\} \tag{8}$$

Eq. (8) can be rewritten as

$$[K_m]\{q\} = \{f_m\} \tag{9}$$

where

$$[K_m] = [B_m]^T[S_m][B_m] \tag{10a}$$

$$[f_m] = [B_m]^T\{M\} \tag{10b}$$

As specific illustration,

$$\{f\}^T_{\text{col}} = \{V_i,\ M_i,\ N_i,\ V_j,\ M_j,\ N_j\} \tag{10c}$$

$$\{f\}^T_{\text{beam}} = \{V_i,\ M_i,\ V_j,\ M_j\} \tag{10d}$$

Critical issue in modelling and subsequent nonlinear analysis of RC frame structures is to evaluate the stiffness matrix $[S_m]$ of a member at a given load level. As mentioned earlier, uneven cracking and yielding in a member will induce substantial reduction in the member stiffness at near-ultimate load levels. In the present formulation, both concentrated and distributed cracking are considered in the evaluation of member stiffness matrix. For mathematical representation, it is convenient to calculate the additional flexibility of a member due to crack formation or yielding rather than to calculate the reduction in stiffness of the member (Kunnath *et al.* 1991, Cipollina *et al.* 1995). Therefore, in the following sections, member characterization is done in terms of its flexibility matrix which can be finally inverted and used for further calculations.

2.2. Member idealisation

Concrete cracking and reinforcement yielding are considered to be responsible for nonlinear

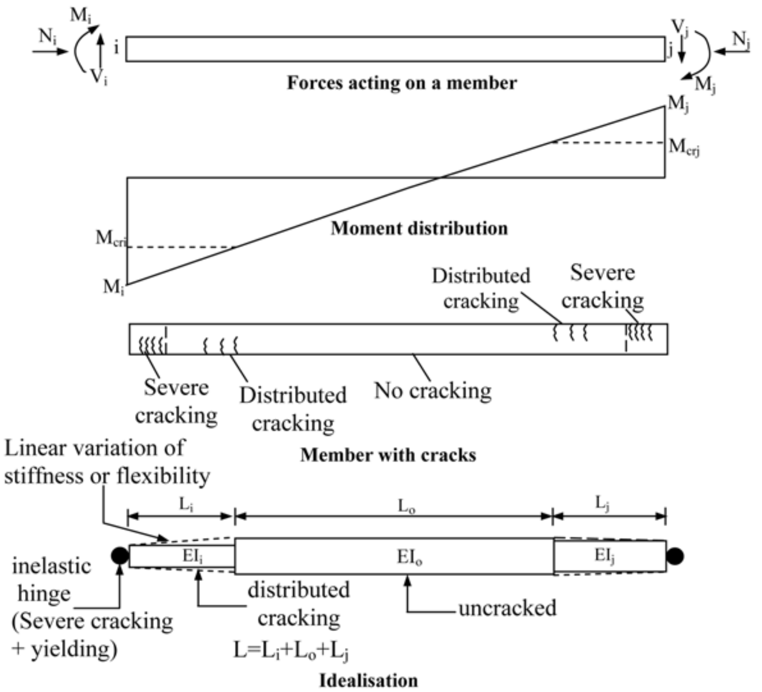

Fig. 4 Representation of a member with non-prismatic section and inelastic hinges

behaviour of the member induced by stiffness degradation. Concrete cracking primarily results in unequal reduction of section stiffness over the cracked length resulting in, transfer of the load from concrete to reinforcement. In the case of reinforcement yielding, a plastic hinge is likely to develop over a short distance at a section due to which, primarily, the member will undergo additional deformation besides showing concentrated cracking. In the present formulation, the concepts of CDM are used to quantify the effects of cracking and plasticity theory coupled with effective stress concept of CDM is employed to model the effects of reinforcement yielding.

2.3. Hinge model

A frame member is shown in Fig. 4 along with the variation of moment in it. Moment demand at the member ends are denoted by M_i and M_j while the corresponding cracking moment capacity of the section is denoted by M_{cri} and M_{crj}. It is shown that flexural cracks appear at those sections in which the moment demand exceeds the cracking capacity. Further, it is considered that cracks form in concentrated manner at sections very close to the ends (Cipollina *et al.* 1995). On increasing the moment demand, reinforcement at those end sections showing concentrated cracking are likely to yield. It is proposed that concentrated cracking and reinforcement yielding introduce nonlinearity in the response behaviour of RC frames. As per lumped plasticity approach, therefore, a plastic hinge is modelled at member ends to equivalently represent the nonlinear effects.

As shown in Fig. 4, it is considered that a short span of the member denoted by L_i and L_j at ends i and j suffer from distributed cracks. Therefore, the member idealization shows that those sections containing distributed cracks are modelled with reduced stiffness of $EI_i<EI_o$ and $EI_j<EI_o$. In effect

the frame member is idealized as non prismatic as shown in Fig. 4. The reduced stiffness corresponding to cracked section can be obtained from a typical trilinear moment-curvature (He *et al.* 2007) relation of a flexural member.

Based on the discussions on the member behaviour w.r.t. cracking, i.e., concentrated and distributed, it is proposed that the deformations of a frame member are split into components as

$$\{Q_m\} = \{Q^p\} + \{Q^d\} + \{Q^{eii}\} + \{Q^{eio}\} + \{Q^{eij}\} \tag{11}$$

where

$\{Q^p\}$ contains plastic rotations of inelastic hinges

$\{Q^d\}$ contains inelastic deformation due to severe (concentrated) cracking at end sections

$\{Q^{eii}\}, \{Q^{eio}\}$ and $\{Q^{eij}\}$ contain elastic deformations of either uncracked section or sections that contain distributed cracks.

Among these, $\{Q^p\}$ and $\{Q^d\}$ are modelled by the inelastic hinges. Plastic deformation denoted by the vector $\{Q^p\}$ is proposed to be evaluated by making use of plasticity theory combined with effective stress concept. Therefore, it does not confine to the conventional force-deformation relation represented using flexibility or stiffness of the member. The deformation component due to damage, $\{Q^d\}$, is proposed to be evaluated by integrating the conventional force-deformation relation embedded with damage variables to represent the gradual degradation in the member stiffness due to concentrated cracking. The damage variables are defined within the frame work of CDM. In that case, the deformation component due to damage can be expressed as

$$\{Q^d\} = [F^d(D)]\{M\} \tag{12}$$

where $[F^d(D)]$ can be interpreted as additional flexibility of the member that arise due to concentrated cracking at the member ends. This additional flexibility is computed as a function of damage variables associated with the inelastic hinges modelled in the member. It is ensured that the additional flexibility expressed in any form reduces to zero for uncracked state (d_i=0) of a member. A simple form of the additional flexibility matrix can be written as (Florez-Lopez 1998)

$$[F_d] = \begin{bmatrix} f_{11} & 0 & 0 \\ & \dfrac{f_{22}}{1-d_i} & 0 \\ \text{symm} & & \dfrac{f_{33}}{1-d_j} \end{bmatrix} \tag{13}$$

where d_i and d_j are damage variables associated with inelastic hinges at ends i and j; f_{11}, f_{22} and f_{33} are elastic flexibility coefficients given as

$$f_{11} = \frac{L}{AE},\ f_{22} = f_{33} = \frac{L}{3EI_o}$$

and AE is axial stiffness of the section; EI_o is the stiffness coefficient of the member in uncracked state. While defining the coefficients of the additional flexibility matrix, it is implied that the member degrades only in terms of its flexural rigidity due to flexural cracking while its axial stiffness remains intact. It is also assumed that the frame members have sufficient capacity to resist the shear demand on the section without undergoing degradation.

2.4. Elastic member

For a given value of moment demand, a frame member is expected to show concentrated cracking at ends and distributed cracking over a finite region adjacent to concentrated cracking. The extent of distributed cracking is decided basically by moment distribution in the member, with cracking moment capacity of the section as delimiter. In the present model, it is proposed to include a non-prismatic member in the idealisation to take care of the distributed cracking. Precisely the three segment model shown in Fig. 3 is adopted to model the elastic behaviour of the member. Length of the individual segments is found according to the moment demand on the member. It is taken care of in the implementation to dynamically vary the length of the segments to obtain equivalent stiffness of the non-prismatic elastic member.

In the interest to have an equivalent representation of the non-prismatic member, a simplification is introduced as

$$\{Q^{eii}\}+\{Q^{eio}\}+\{Q^{eij}\} = [F^e(M)]\{M\} \tag{14}$$

where

$[F^e(M)]$ is the equivalent flexibility matrix of the non-prismatic member given as $\begin{bmatrix} f_{ii} & f_{ij} \\ f_{ji} & f_{jj} \end{bmatrix}$.

As an alternative, equivalent flexibility of the segmented member can be modelled to be linearly varying as shown by dotted lines in Fig. 4. The flexibility coefficients for linear variation of member flexibility are (Lobo 1994)

$$\begin{aligned} f_{ii} &= C.f'_{ii} \\ f_{ij} = f_{ji} &= C.f'_{ij} \\ f_{jj} &= C.f'_{jj} \end{aligned} \tag{15a}$$

where

$$C = \frac{L}{12EI_oEI_iEI_j} \tag{16a}$$

$$f'_{ii} = 4EI_iEI_j + (EI_o - EI_i)EI_j(6\alpha_i - 4\alpha_i^2 + \alpha_i^3) + (EI_o - EI_j)EI_i\alpha_j^3 \tag{16b}$$

$$f'_{ij} = -2EI_iEI_j - (EI_o - EI_i)EI_j(2\alpha_i^2 - \alpha_i^3) - (EI_o - EI_j)EI_i(2\alpha_j^2 - \alpha_j^3) \tag{16c}$$

$$f'_{jj} = 4EI_iEI_j + (EI_o - EI_i)EI_j\alpha_i^3 + (EI_o - EI_j)EI_i(6\alpha_j - 4\alpha_i^2 + \alpha_j^3) \tag{16d}$$

$$\alpha_i = \frac{L_i}{L};\ \alpha_j = \frac{L_j}{L}$$

EI_i and EI_j are flexural rigidities of the end section of the member extending to a length of L_i and L_j, respectively, as shown in Fig. 4. EI_o is the flexural rigidity of the uncracked mid-section of the member.

The variable lengths α_i, α_j can be expressed as a function of the imposed moment and cracking moment capacity of either ends of a member. For example, for double curvature deformation of a member

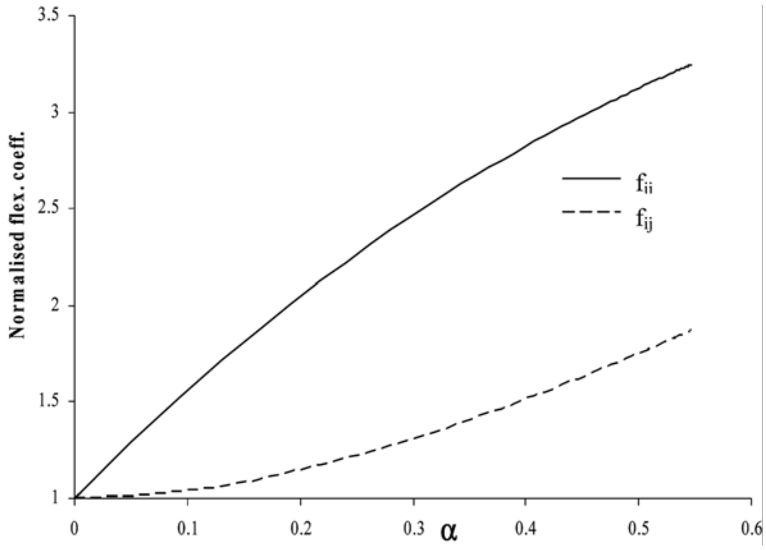

Fig. 5 Variation of flexibility coefficient w.r.t. length of cracked segment

$$\alpha_i = \frac{M_i - M_{cri}}{M_i - M_j} \tag{16e}$$

$$\alpha_j = \frac{M_j - M_{crj}}{M_j - M_i} \tag{16f}$$

where the variables are defined in Fig. 4. A similar set of expression can be defined for single curvature deformation of a member.

Variation in the normalized values of flexibility coefficients with the length of segment having distributed cracks is shown in Fig. 5. The coefficient values are normalized w.r.t. that for α=0. It is seen that the flexibility coefficient of a member increases by a maximum of about 3 times under extreme condition of cracks distributed over one half of the member.

Eq. (11) can be rewritten using Eqs. (12) and (14) as

$$\{Q_m\} = \{Q^p\} + ([F^d(D)] + [F^e(M)])\{M\} \tag{17}$$

$$\{Q_m\} = \{Q^p\} + [F_m(D, M)]\{M\} \tag{18}$$

where

$$[F_m(D, M)] = [F^d(D)] + [F^e(M)] \tag{19}$$

Eq. (18) can be rearranged in terms of stiffness as

$$[S_m(D, M)]\{Q_m\} = \{M\} + [S_m(D,\ M)]\{Q^p\} \tag{20}$$

where

$$[S_m(D, M)] = [F_m(D, M)]^{-1} \tag{21}$$

Now using Eq. (20) instead of Eq. (4) and following similar mathematical steps, Eq. (9) can be rewritten as

$$[K_m(D,M)]\{q\} = \{f_m\} + [B_m]^T[S_m(D,M)]\{Q^p\} \tag{22}$$

where

$$[K_m(D,M)] = [B_m]^T[S_m(D,M)][B_m] \tag{23}$$

The second term on RHS of Eq. (22) is analogous to force as it is product of stiffness and rotations. Since the term contains plastic rotations, it can be termed as plastic force vector $\{f^p\}$. Therefore Eq. (22) is written as

$$[K_m(D,M)]\{q\} = \{f_m\} + \{f^p\} \tag{24}$$

Eq. (24) is applicable with appropriate terms for each column and beam of the frame structure. This equation represents equilibrium state of a member by equating the internal and external forces associated with a member. After evaluating the stiffness matrix of each member, it is possible to assemble them to form the global stiffness matrix of the structure. This can be expressed as

$$\sum_{m=1}^{M}[K_m(D,M)]\{r\} = \sum_{m=1}^{M}\{f_m\} + \sum_{m=1}^{M}[B_m]^T[S_m(D,M)]\{Q^p\} \tag{25}$$

Eq. (25) can be written simply as

$$\mathbf{Kr} = \mathbf{f} \tag{26}$$

where **K** is the global stiffness matrix; **f** is the global force vector and **r** is defined in Eq. (3).

Equilibrium of the frame structure expressed in Eq. (26) contain two additional variables, viz., damage variable (d) and plastic rotation (ϕ_{ip}) for each active inelastic hinge in the idealization. Damage evolution law and plastic function are used to numerically evaluate their values in the solution of nonlinear equations.

2.5. Damage function

Damage function serves the purpose to model the effect of concrete cracking on member stiffness. It is defined in terms of scalar damage variable and operate upon the virtual inelastic hinges modelled in a frame member. Damage function is used to define the state and evolution of damage at a hinge. Based on Griffith's criterion, damage function is expressed as

$$g_i = G_i - R_i \tag{27}$$

where G_i is the strain energy release rate of a hinge conjugated with damage variables, and R_i defines the resistance of the member to cracking. Member resistance to flexural cracking is evaluated through experiment as (Cipollina *et al.* 1995)

$$R(d_i) = G_{cri} + q_i\frac{\ln(1-d_i)}{(1-d_i)} \tag{28}$$

where

G_{cri} and q_i are member-dependent constants.

Therefore, the damage function (g_i) can be expressed as

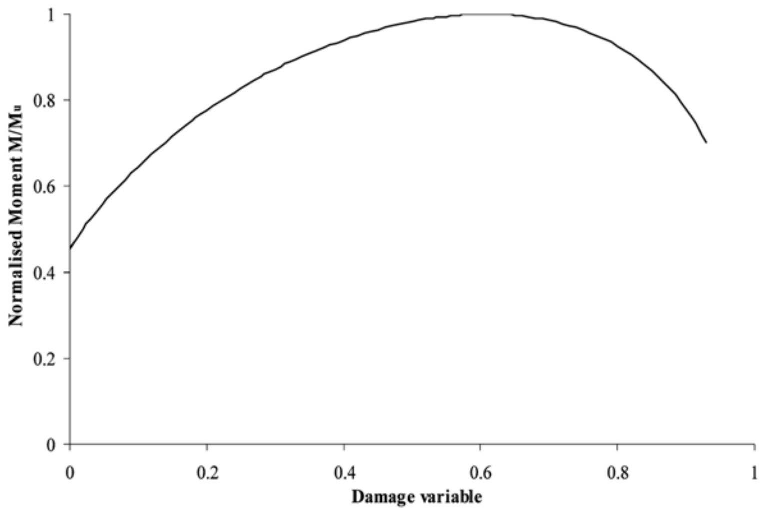

Fig. 6 Damage function

$$g_i = G_i - \left(G_{cri} + q_i \frac{\ln(1-d_i)}{(1-d_i)} \right) \tag{29}$$

Conditions for damage growth are given as

$$\text{Damage stagnant or no damage: } d_i < 0 \text{ for } g_i < 0 \text{ and } \dot{g}_i < 0 \tag{30a}$$

$$\text{Growth of damage: } d_i > 0 \text{ for } g_i = 0 \text{ and } \dot{g}_i = 0 \tag{30b}$$

where $\dot{g}_i$ is the gradient of the damage function w.r.t. loading rate.

Member-dependent constants are evaluated based on the conditions suggested by Perdomo *et al.* (1999). The constant G_{cri} is a function of cracking moment capacity of the section and

$$G_i = \frac{M_i^2 L}{6\mathrm{EI}(1-d_i)^2} \tag{31}$$

where EI is the uncracked stiffness coefficient of the section. Eq. (29) describes nonlinear relation between the demand and the capacity of a section to cracking in terms of a non-dimensional damage variable and is plotted in Fig. 6 for assumed variation in the moment at a hinge. It can be seen that the damage function is capable to reproduce the softening behaviour of a RC flexural member.

2.6. Plastic function

A plastic function to describe the evolution of plastic rotation at a hinge with moment is used in the present study (Cipollina *et al.* 1995). It is expressed as

$$f_i = \left| \frac{M_i}{1-d_i} - C_{1i}\phi_{ip} \right| - C_{2i} \tag{32}$$

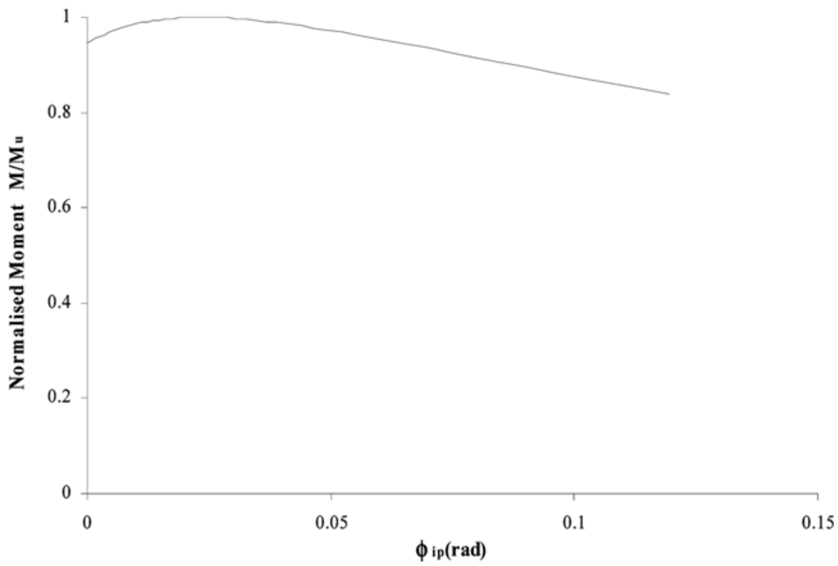

Fig. 7 Plastic function

where

C_{1i} is a constant to denote the hardening behaviour of the section,

ϕ_{ip} is the plastic rotation of the hinge and

$C_{2\mathrm{i}}$ is a constant proportional to yield moment of the section.

Together with this, the condition for plastic evolution is specified by

$$f_i = 0,\ \dot{\phi}_{ip} > 0 \tag{33a}$$

and stagnation in plastic growth is identified by

$$f_i < 0,\ \dot{\phi}_{ip} = 0 \tag{33b}$$

where $\dot{\phi}_{ip}$ is the rate of growth of plastic rotation w.r.t. loading rate. Again, the member-dependent constants in plastic function are evaluated by following Perdomo *et al.* (1999). Plastic function is plotted in Fig. 7 for consistent values of moment and damage variable obtained by using Eq. (29). Plastic function satisfies the general condition w.r.t. initiation and growth of plastic rotation at a hinge.

2.7. Numerical implementation

Numerical implementation of the proposed formulation is carried out by following global-local strategy.

2.7.1. Global

At global level, structure is treated to know its deformations. Kinematic variables defined in Eq. 3 are considered as unknowns. Equilibrium equations of the structure (Eq. 26) are solved for these unknowns.

2.7.2. Local

At local level, each member is treated to solve for the associated variables. As a general case, each frame member consists of 9 unknowns, namely, 2 moments (M_i and M_j), axial load (P_i), 2 damage variables (d_i and d_j), 2 plastic rotations (ϕ_{ip} and ϕ_{jp}) and 2 variables for length of cracked segments (α_i and α_j). Three equations due to member equilibrium (Eq. 20), two equations due to damage function (Eq. 29), two equations due to plastic function (Eq. 32) and two equations that express the length of cracked segment (Eqs. 16e and 16f) are used to solve for the 9 local variables. These form a set of nonlinear equations and Newton's technique (Johnston 1982) is used to solve for the local variables.

External displacement load is applied incrementally in steps. Solution is carried out in iterative manner to ensure convergence at both global and local levels. Basic assumption in the formulation is that the nonlinear behaviour of the structure is due to concrete cracking and yielding of reinforcement. Therefore, geometric nonlinearity is accounted in a simple manner by P-Δ effect. Numerical implementation is done in such a way that only following details are required to describe the structure for analysis:

i) Geometry and support details of the structure and external loads
ii) Cracking, yielding and ultimate moment capacities of the sections

3. Numerical investigations

3.1. Example 1: RC column subjected to constant axial load

Numerical performance of the proposed model is carried out by analyzing a RC column subjected to constant axial load. The column is additionally loaded through lateral displacement applied at

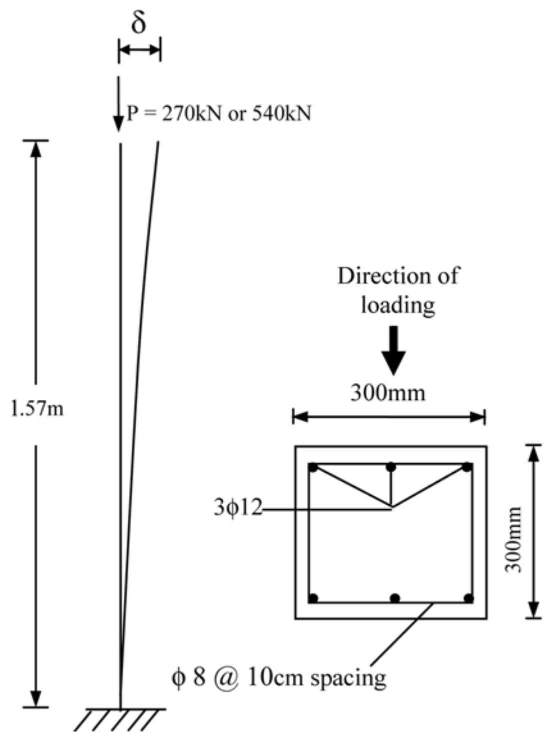

Fig. 8 Reinforced concrete column subjected to axial load and lateral displacement

Table 1 Model parameters and their values

Model parameter	Value	
	270 kN	540 kN
Cracking moment, M_{cr} (kNm)	27.0	40.5
Initial stiffness, EI_{uncr} (kNm^2)	1.5×10^4	1.5×10^4
Yielding moment, M_y (kNm)	59.4	91.2
Yielding curvature, χ_y (rad/m)	1.1×10^{-2}	1.7×10^{-2}
Ultimate moment, M_u (kNm)	64.6	93.4
Plastic rotation capacity, $\phi_{pl}^{u}(rad)$	1.3×10^{-2}	0.8×10^{-2}

top. Nonlinear behaviour of the column is studied using the variation in base shear against the applied lateral displacement. Fig. 8 shows the basic details of the problem. The column is experimentally studied (Verderame *et al.* 2008) for two different axial loads, viz., 270 kN and 540 kN. Cylindrical compressive strength of concrete is 25 MPa. Longitudinal steel has yield stress of 355 MPa and ultimate stress of 470 MPa with 27% failure strain. Cracking, yielding and ultimate moment capacities and the corresponding displacement capacities of the column calculated based on the strength characteristics of concrete and steel are used in the analysis. Numerical values of the model parameters are available in Table 1. Nonlinear quasi-static analysis has been carried out with control on displacement which is applied in steps till the damage variable attained a reasonably large value (≈ 0.8).

3.1.1. Axial load : 270 kN

For 270 kN axial load, experimental results are available for 2 trials and the response from both the trials is reproduced in Fig. 9. Computed response has also been plotted in Fig. 9 for comparison. Nonlinear response of the column is studied for about 6% drift. As an additional case, the column is analysed without considering the effect of distributed cracking. This amounts to modelling the elastic behaviour of the column as a prismatic member with stiffness equal to that of uncracked section. This case is analysed in order to clearly identify the effect of non-prismatic elastic model towards the overall response behaviour of the structure. It may be noted the specific contribution of the elastic member idealization used in the present work also can be brought out by this comparison. From Fig. 9, it can be seen that the response from present study based on non-prismatic elastic member idealization matches better with that due to experiment. Another clear inference from Fig. 9 is that the method used to account for distributed cracking in a frame member is necessary and also adequate to predict the nonlinear response between post-racking and pre-yielding stage. Differences between the predicted and measured responses shall be treated to be insignificant in the background of the differences between the two trials of experiment.

The ultimate base shear capacity of the column has been predicted as 41.2 kN in the analysis while the same has been reported as 41.7 kN in experiment. Intermediate results from analysis indicate that first crack appear in the bottom of the column for the base shear value of 17.1 kN. This is known from the load step at which the damage variable attained positive value. Reinforcement at the same location is found to yield when the base shear value reached 38 kN and the damage variable attained a value of 0.37. The member started showing plastic rotation after this and consequently non-zero plastic forces were included in the governing equilibrium equations.

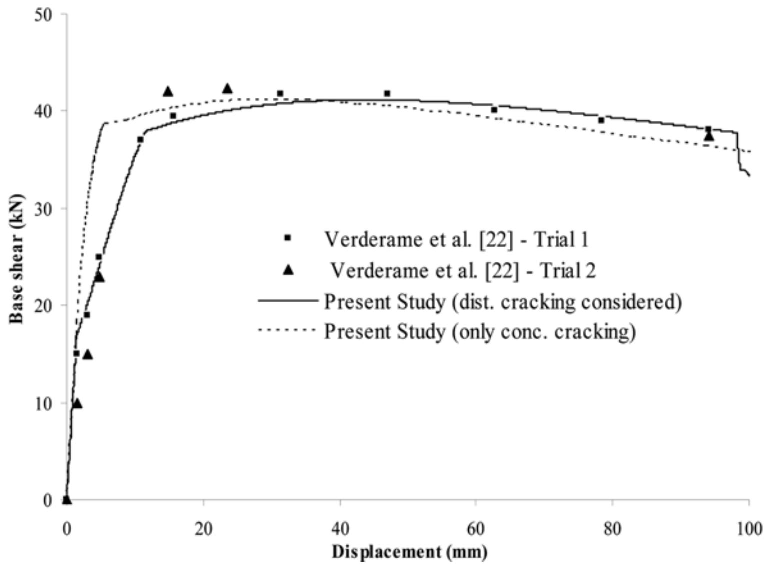

Fig. 9 Variation of base shear with top displacement for 270 kN axial load

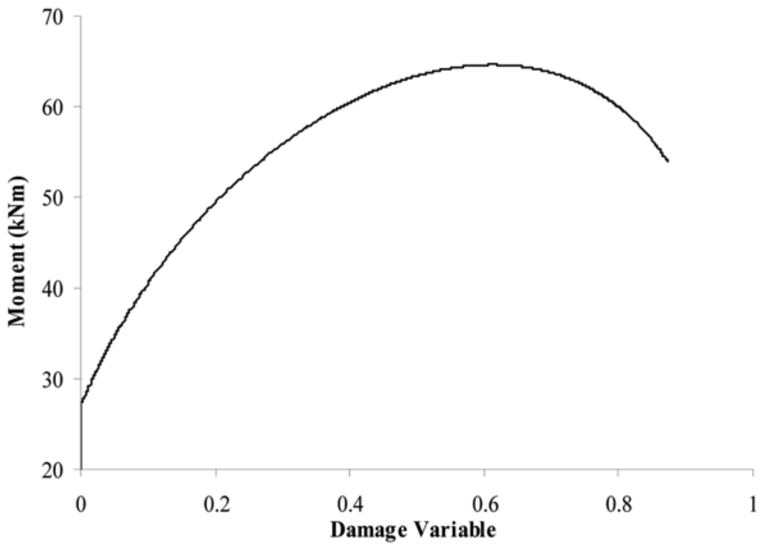

Fig. 10 Variation of damage variable with moment at base of the column

Detailed results of diagnostic nature are studied as part of an exercise to gain an insight on the performance of the model. At the same time, this helps to verify the numerical implementation of the formulation and the solution procedure. Variation of local variables in the formulation with respect to the global response of the column has been plotted in Figs. 10-15.

Variation of base moment with corresponding value of damage variable and plastic rotation has been plotted in Figs. 10 and 11. The plot is found to be reproduction of damage function and plastic function available. As per Fig. 10, the column starts experiencing damage by cracking when the base moment reaches 27 kNm; damage value at the base of the column is about 0.6 when the moment value is equal to 64.6 kNm. These verify with respect to the values of the model parameters listed in Table 1. Similarly plastic deformation arises in the section for the moment value of about 60 kNm which is equal to its yield moment (Fig. 11). The same figure also indicates that plastic rotation is equal to the capacity of the section for the moment equal to its ultimate moment.

Moment generation at the base due to lateral displacement at top depends on the residual stiffness of the column. Fig. 12 shows the increase in the base moment due to applied lateral displacement loading. The plot shows the reduction of stiffness of the section throughout the loading range. This also confirms with the trilinear moment curvature relation used to describe the flexural behaviour of the section and the subsequent influence of damage variable and plastic rotation. The drastic reduction in the section stiffness after reinforcement yielding could be traced in the analysis based on the formulation.

Evolution of damage and plastic rotation at the base of the column with the applied displacement is shown in Figs. 13 and 14. It can be seen that damage evolution at the base is steep till reinforcement yields and thereafter it becomes relatively slow (Fig. 13). Accelerated accumulation of plastic rotation with applied displacement can be seen from Fig. 14. This presents, however, only minor change w.r.t. the change in slope of the curve.

The formulation presents an improvement in terms of accounting the effect of distributed cracks in a frame member. Therefore, to verify its direct influence, variation of member length expected to experience distributed cracking due to moment exceedance is plotted in Fig. 15 w.r.t. damage variable and in Fig. 16 w.r.t. the applied displacement. Both the figures indicate that the segment containing distributed cracks grow at faster rate initially to finally remain stationary at about 60% of the height of the column. This is again governed by the variation of the moment along the column. This also validates the expected behaviour that after yielding of reinforcement, flexural cracks mostly appear concentrated in the short length over which the reinforcement is assumed to yield (theoretical length of plastic hinge).

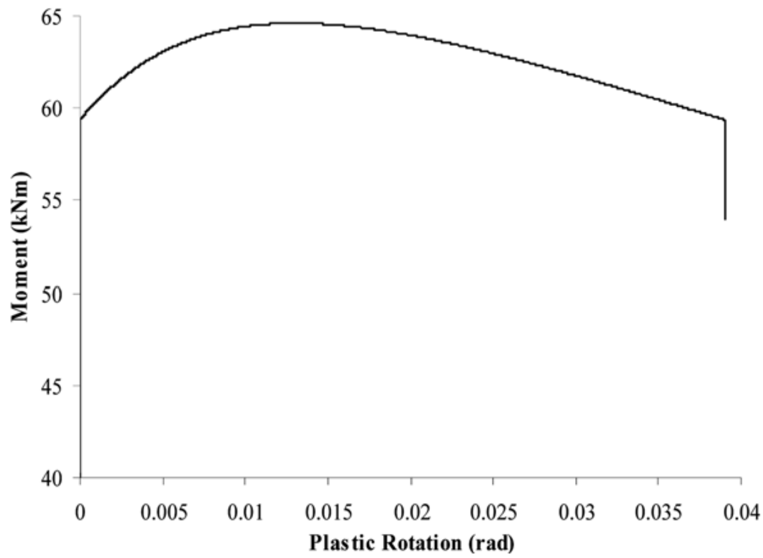

Fig. 11 Variation of plastic rotation with moment at base of the column

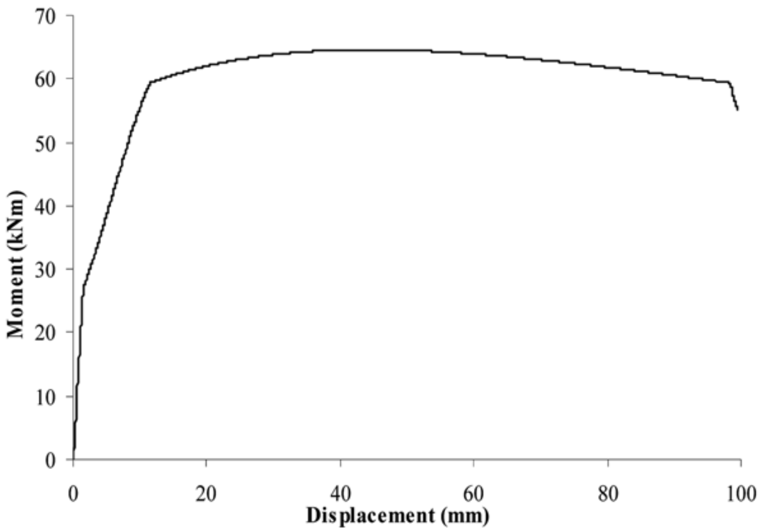

Fig. 12 Variation of top displacement with base moment of the column

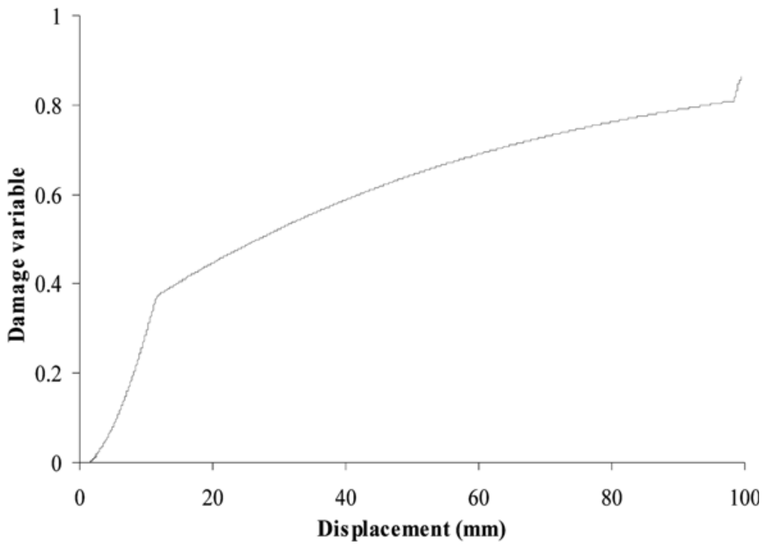

Fig. 13 Damage evolution at base of the column with applied displacement

3.1.2. Axial load : 540 kN

Variation of base shear with applied displacement at top of column is shown in Fig. 17. For this case also, the column response is studied for about 6% drift. Comparison of the computed results with that due to experiment enables to reach similar conclusion as like in the case of column subjected to 270 kN axial load. Ultimate shear capacity of the column has been computed as 59.5 kN while this value is found in experiment as 63.2 kN. In the analysis, the column was found to exhibit cracking and yielding while the base shear values were 25.7 kN and 58.1 kN, respectively. Damage variable value corresponding to initiation of yielding is found to be 0.48.

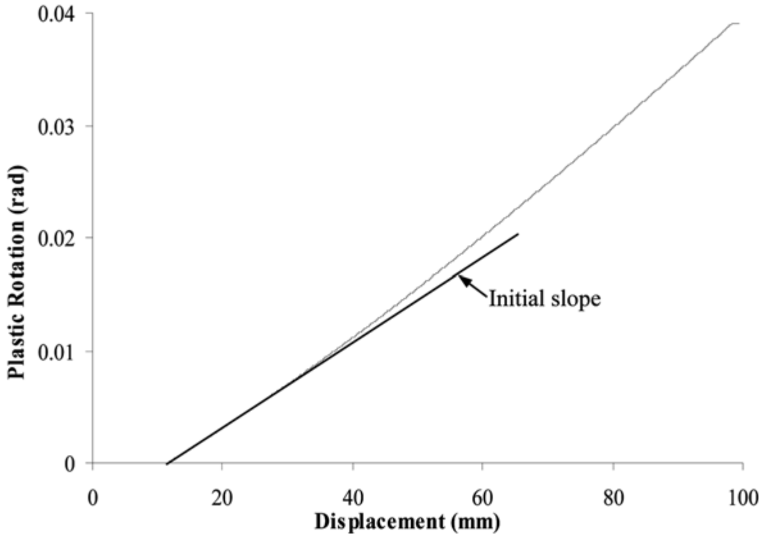

Fig. 14 Evolution of plastic rotation at base of the column with applied displacement

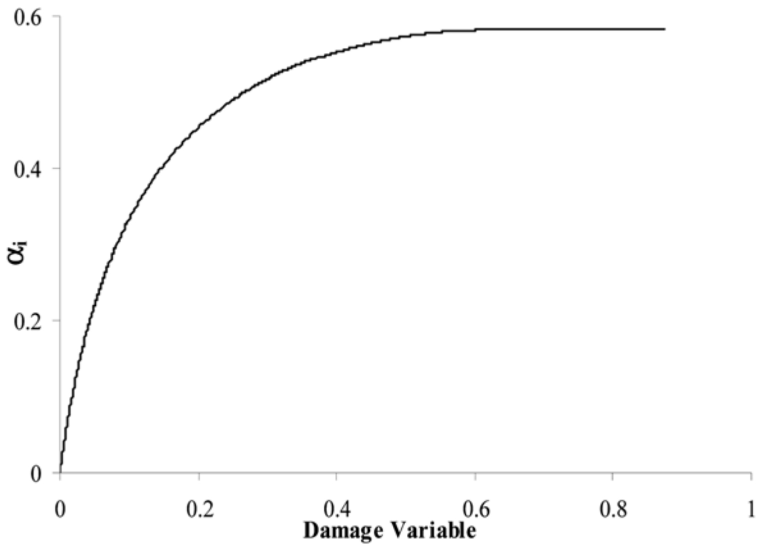

Fig. 15 Variation of segment containing distributed cracks with damage variable

3.2. Example 2: Wide two-storey RC frame

A wide two-storey RC frame is analysed for its quasi-static response due to monotonic lateral loading with inverted triangle distribution. Fig. 18 gives the major dimensions of the frame and the section details of column and beam. Faleiro *et al.* (2008) have analysed the frame for the same loading using a plastic-damage model that includes an exponential softening function. Proposed model is different from that used in the reference in terms of the logarithmic damage function and elastic member definition. Characteristic features of the frame that need to be noted are:

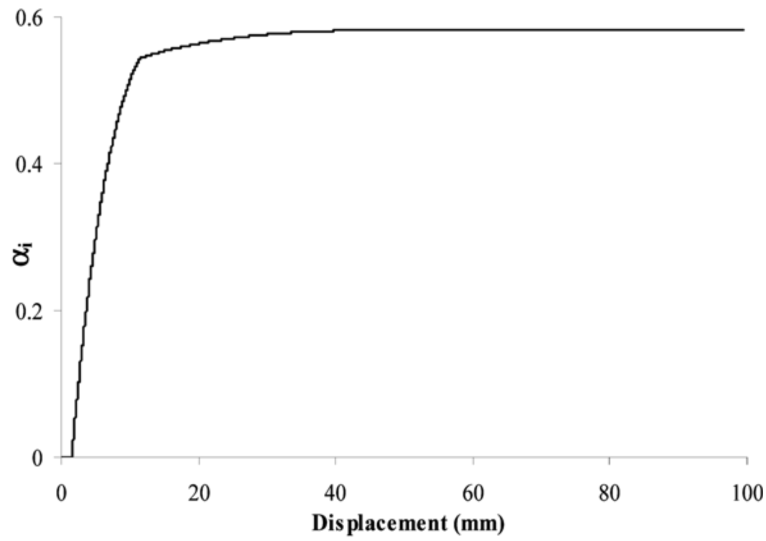

Fig. 16 Growth of segment containing distributed cracks with applied displacement

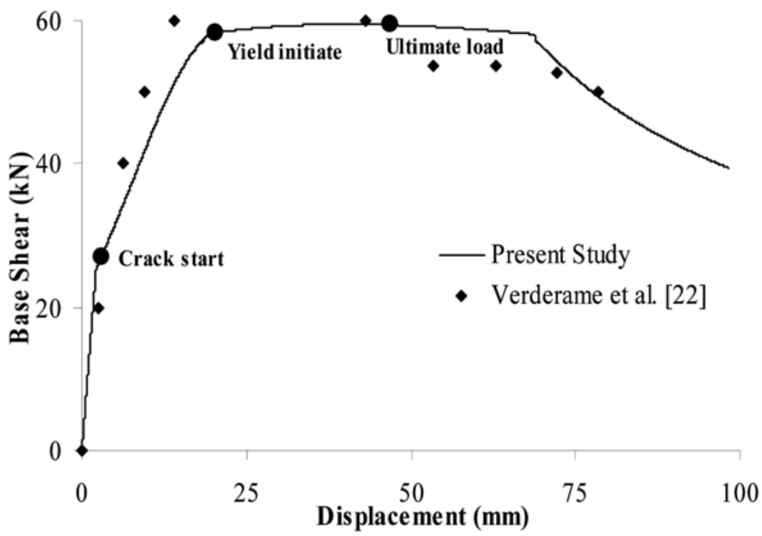

Fig. 17 Variation of base shear with top displacement for 540 kN axial load

- Beam spans are considerably large for the section and the percentage reinforcement.
- Height to width of the frame is 1.0 which means columns are stiffer than beams.
- Beams are reinforced unsymmetrically, i.e., top and bottom reinforcement in the beams is different. Due to this, the beams have different capacities for hogging and sagging moments and undergo different degrees of damage for same moment demand.

Compressive strength of cylindrical specimens of concrete is 25 MPa. Yield strength of the main reinforcement is taken as 420 MPa. Reinforcement is considered to have a clear cover of 40 mm. Capacities of the column and beam sections computed based on the above strength characteristics are used to model the members in the analysis.

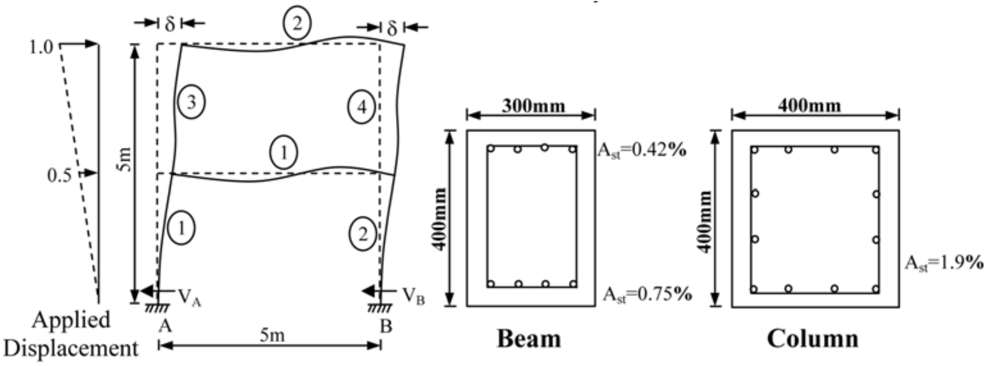

Fig. 18 Details of a wide two-storey frame

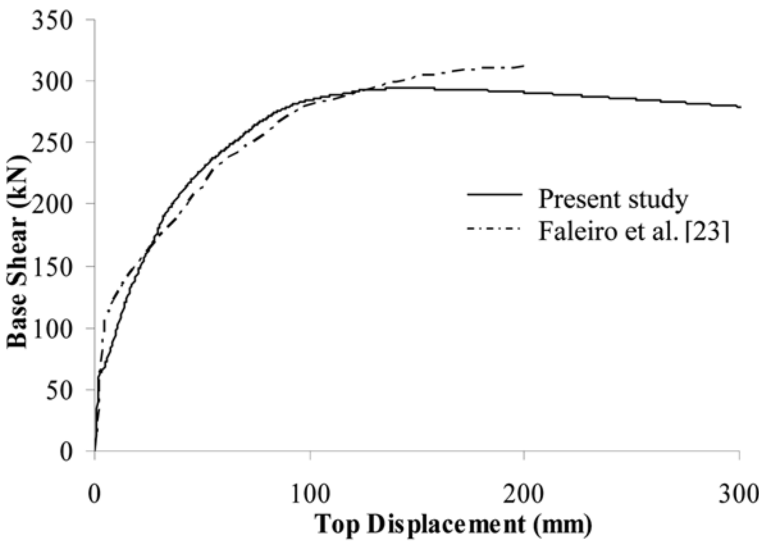

Fig. 19 Variation of base shear with second storey displacement

Fig. 19 gives the variation of base shear w.r.t. displacement at second storey level of the frame. Base shear is computed as sum of the shear in the columns of the first storey. Variation of base shear is available in literature only for 200 mm of second storey displacement which presents almost up to ultimate strength of the frame. In the absence of the softening response of the frame, it is not able to assess the capability of the method used in the reference to reproduce the softening response. Present analysis was, however, carried out up to 300 mm (6% drift) mainly to verify the reproduction of softening response of the frame. It can easily be verified from Fig. 19 that the results computed in present analysis matches favourably with the reference value for the entire range for which the results are available. In addition, the results of present analysis prove the capability of the model to reproduce the softening response of the frame. Ultimate strength of the frame predicted by the present analysis is 294.2 kN (147mm top displacement, 3% drift) while the reference value can be taken reasonably as 310 kN (final value on the results plot; 200 mm top displacement, 4% drift).

4. Conclusions

Simple formulation accompanied by a model based on inelastic hinges and elastic member is presented in this paper for nonlinear analysis of reinforced concrete frame structures. Continuum damage mechanics principles are used to define the degrading deformation behaviour of the inelastic hinges. Basis of the formulation is behaviour of a typical RC frame member is governed by flexure. Nonlinear structural behaviour due to concentrated cracking and reinforced yielding are modelled by using suitable moment-deformation relation for the inelastic hinges. An isotropic damage variable based on CDM is used to compute the additional flexibility of the frame member due to cracking. Characteristic feature of the model is to include the distributed cracks in a member by modelling the member with non-prismatic sections. Moment variation in the member is taken as the basis to decide on the features of non-prismatic member. Besides these, the plastic deformation due to reinforcement yielding is also accounted by applying necessary conditions.

Quasi-static analysis of a RC column subjected to constant axial load and a wide two-storey RC frame has been carried out to evaluate the performance of the model. The lateral deformation behaviour of the structures for controlled displacement applied at the top has been evaluated for different axial loads. Computed response has been matched with those due to experiment over the entire loading range. Comparison of the results confirms excellent performance of the model particularly in the range between cracking and yielding. Results of diagnostic nature for the column are presented in the paper with the aim to derive an insight into the behaviour of the structure. The formulation is considered to be superior to other similar models due to its non-dependence on constants to define the degradation of a member besides requiring only minimum input.

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