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Shape optimization of steel reinforced concrete beams

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Abstract. Steel reinforced concrete is perhaps the most versatile and widely used construction material. The versatility is attributed to mouldability of concrete to any conceivable shape. The inherent property of cracking of concrete is the reason for its low tensile strength and hence the design approach of RCC sections in flexure adopts the cracked section theory where in concrete in tension zone is ignored. Means, modes and methods of exploitation of concrete strength by conceiving shapes other than rectangular whereby ineffective concrete in tension zone is reduced and incorporated in compression zone where it is effective needs consideration. Shape optimization of beams is attempted in this analytical investigation employing Sequential Unconstrained Minimization Technique (SUMT). The results clearly show that trapezoidal beams happen to be less costlier than their rectangular counterparts, their usage needs serious reconsideration by the designers.

Keywords: reinforced concrete; simply supported beams; uniformly distributed load; limit state design; shape; optimization

1 Introduction

Structures are built with an intended end use for a specified utility period. The functional requirement is to serve the purpose and the structural requirement constitutes stability, strength, safety, serviceability and durability. Satisfaction of these requirements at affordable costs is the goal of structural optimization. Realization of cost effective structures heavily relies on factors like progress made in material sciences, knowledge of behavior of materials and structural systems, analytical tools for assessment of strength, safety and serviceability, construction practices and construction economy.

Structures in the context of Civil and Structural Engineering are arrangements of load resisting elements to suit specific needs. Tension, compression, flexural, membrane elements and others are assembled to constitute the geometry of the structure and the assemblage is called the configuration – slabs, beams, columns, plates, shells, struts, ties, cables, membranes are the various elements in use individually and in combination. Selection of right element and combination depends on simplicity, efficacy, adaptability and aesthetic quality. Increased appreciation of basic forms essentially comes from better understanding of functional and structural requirements and structural behavior of basic forms.

Accomplishment of structural economy is possible by an integrated approach. As functional and structural requirements to be satisfied are dependent on a wide and varied range of factors, the

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techniques of operation research come handy in realization of optimal structures. The objective function to be minimized or maximized is chosen as the cost or the benefits respectively. The various factors affecting cost are taken as the design variables. The functional and structural requirements are considered as the constraints. The optimization-programming problem is formulated and the solution is sought from among the vast range of techniques available that suits the problem the best.

Teonis and Lefaiure (1975) credit Galileo Galilei as the initiator of shape optimization, for it was Galileo who made the famous proposition that "it would be a fine thing if one could discover the proper shape to give to an object in order to make it equally resistant at every point". Much work has been done since then and the search is still on, as there is no single way to address the wide and varied range of problems encountered in structural engineering. The fact that there exists a unique geometry for the structure of absolute minimum weight under a specific loading arrangement was first suggested by Michell (1904). The optimal structures are built up of orthogonal nets of pin jointed members shaped like slip-lines. Richards and Chan (1966) have reviewed and confirmed Michell's theory. Though Michell's theory gives an inverse design method it cannot include many design considerations and hence finds little application. Nevertheless, Michell's work is regarded as the beginning of shape optimization because it was recognized for the first time that shape should be a fundamental variable in designing an efficient structure. A method of design that finds best shape without investigating all possible shapes was first attempted by Dorn, *et al.* (1964) employing linear programming technique. The results were in the range from the interesting to the unexpected. As objective functions and constraints in structural optimization generally are non-linear, it was accepted that clumsy liberalizations to include them in linear programming was of no much help. Application of dynamic programming, a method of non-linear programming technique was suggested by Bellman (1957) and applied to pin jointed frameworks by Porter-Goff (1968). Failure to adequately include design considerations makes this technique generate solutions that are not always optimal. Palmer (1971) has used dynamic programming for limit state design of beams, to include asymmetry, alternative loading conditions and use of real members instead of artificial concept and has demonstrated the benefit of shape optimization over usage of conventional standard geometry. Geometric programming as a mathematical aid in engineering design was proposed by Zener (1961) and Templeman (1970) showed its utility in structural engineering problems. Though the technique depends on the form of objective function and constraints it can be of use coupled with other methods to find either complete solutions or approximate ones. An approach called Reanalysis combined with sub optimization has been attempted by Lassen (1993) for shape optimization of large three dimensional frameworks. It is assumed that the large frame works are assemblages of small groups of limited number of members and only one active constraint for one member of the group that is critical considered for sub optimization. Sequential quadratic programming technique is employed for sub optimization. Though there is no sound theoretical basis for the assumptions, experience with the technique has indicated that it is robust and consistent, offering optimum solutions as low computational costs. Zimmerman, *et al.* (1993) have suggested application of stochastic optimization models for identification of geometry and failure modes for structures where loads and resistances are to be treated as random variables. An attempt to address axial-force moment interaction as a function of geometry, integrating rigid plastic model with mathematical programming has been made. Calatrava (1981) presents detailed explanation of the subject of shape optimization by defining profiling as selection of geometry on contour and differentiating as dividing the structure to several separate members each performing different functions using materials best suiting members assigned function. He has demonstrated by mathematical programming that three dimensional space frames can be folded first to two dimensions and then into one. With super 3D compass he tried to explain generation of best geometry by movement of the joints in space.

Guerra and Kiousis (2006) have recently presented a novel approach for optimal sizing and reinforcing multi-bay and multi-storey RC structures incorporating optimal stiffness correlation among structural members. Using the Sequential Quadratic Programming (SQP) algorithm implemented in MATLAB's intrinsic optimization function fmincon, the authors have demonstrated the ability of the formulation in achievement of optimal designs. Illustrations have been detailed with parametric studies that control the optimal solution.

Here, it has been envisaged to formulate search algorithms for shape optimization of beams employing Sequential Unconstrained Minimization Technique (SUMT). Since the merit function and constraints in a lot majority of structural optimization problems are non-linear functions of the design variable, it is prudent to accept SUMT as the most appropriate technique for accomplishment of the objective of the problem.

2. Problem formulation and solution

2.1. Design variables

These are grouped as (i) Dimensional variables representing member sizes, area, inertia and sectional modulus (ii) Geometric variables representing coordinates of joints of elements (iii) Mechanical and physical properties like unit weight, elastic modulus, Poisson's ratio. Many times optimization problem is over simplified by fixing the geometry. In the present work it is being attempted to include geometry as a variable.

2.1.1. Design variables for rectangular sections

The objective function and the constraints are the functions of width and the depth, which are

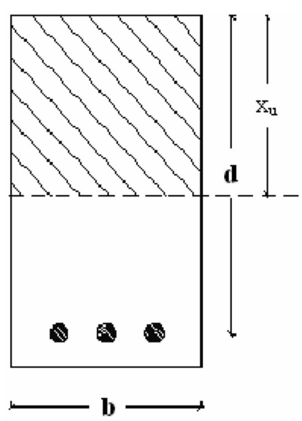

Fig. 1 Cross-section of rectangular beam

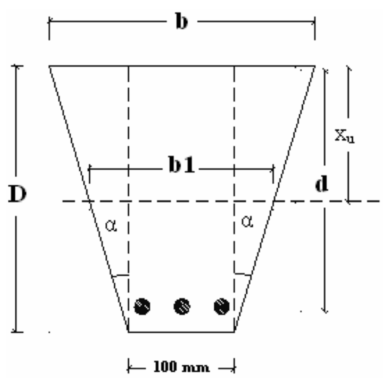

Fig. 2 Cross-section of trapezoidal beam

taken as design variables for the rectangular sections as in Fig. 1.

2.1.2. Design variables for trapezoidal sections

The objective function and the constraints are represented as the functions of depth and alpha (angle), which are taken as design variables for the trapezoidal sections as detailed in Fig. 2.

2.2. Constraints

These are expressed as equalities or inequalities and are sub-divided as (i) Side constraints: physical limitations on the design variables like availability, transportability and fabricability (ii) Behavioral constraints: Limitations on the behavior or performance of the system i.e., restrictions on structural response. They are non-linear functions of design variables.

2.2.1. Constraints for rectangular beams

The following are the constraints considered in the present investigation for optimization of rectangular beams.

1. For the section to be singly reinforced the moment resisting capacity shall be less than the limiting value i.e., $M_u \leq M_{ul}$

$$G(1) = \left(\frac{1.232 M_u}{qbD^2}\right) - 1 \leq 0$$

2. To avoid brittle failure reinforcement should be more than absolute minimum

$$G(2) = \left(\frac{A_{st\min}}{A_{st}}\right) - 1 \leq 0$$

Where $$A_{st\min} = \left(\frac{0.766bD}{f_y A_{st}}\right)$$

3. To ensure that the section is under-reinforced, reinforcement should be less than that for the balanced section.

$$G(3) = \left(\frac{A_{st}}{A_{stb}}\right) - 1 \leq 0$$

Where $$A_{stb} = 0.864\, bD$$

4. From practical considerations

$$b \geq 150; \qquad G(4) = \left(\frac{150}{b}\right) - 1 \leq 0$$

5. From minimum head room requirements

$$D \leq 900; \qquad G(5) = \left(\frac{D}{900}\right) - 1 \leq 0$$

6. From serviceability considerations

$$\frac{5M_uL^2}{48EI} \le \frac{L}{250}; \qquad G(6) = \left(\frac{625M_uL}{EbD^3}\right) - 1 \le 0$$

2.2.2. Constraints for trapezoidal beams

For the optimization of trapezoidal beams, the constraints that have been considered are presented below.

1. To avoid brittle failure reinforcement should be more than absolute minimum

$$G(1) = \left(\frac{A_{st\min}}{A_{st}}\right) - 1 \le 0$$

Where $\quad A_{st\min} = 0.003\ (100\ D + D^2 \tan\alpha)$

2. To ensure that the section is under-reinforced, reinforcement should be less than that for the balanced section.

$$G(2) = \left(\frac{A_{st}}{A_{stb}}\right) - 1 \le 0$$

Where $\quad A_{stb} = (19.77\ D + 0.31\ D^2 \tan\alpha)\ \dfrac{f_{ck}}{f_y}$

3. For the section to be singly reinforced the moment resisting capacity shall be less than the limiting value i.e., $M_u \le M_{ul}$

$$G(3) = \left(\frac{M_u}{M_{ul}}\right) - 1 \le 0$$

Where $\quad M_{ul} = (11.8\ D^2 + 0.19\ D \tan\alpha)\ f_{ck}$

4. For the beam to be trapezoidal he sides should be inclined.

$$\alpha \ge 0^0; \qquad G(4) = -\alpha \le 0$$

5. From practical considerations.

$$\alpha \le 45^0; \qquad G(5) = \alpha - 45^o \le 0$$

6. From serviceability considerations

$$\frac{5M_uL^2}{48EI_{eff}} \le \frac{L}{250} \qquad G(6) = \left(\frac{52.08M_uL}{EI_{gross}}\right) - 1 \le 0$$

where $\quad I_{gross} = \left(\dfrac{30000D^3 + 20D^5\tan^2\alpha + 2400D^4\tan\alpha}{3600 + 36D\tan\alpha}\right) - \tan\alpha\ \dfrac{D^4}{2}$

and $\quad I_{eff} = \dfrac{I_{gross}}{2}$

7. Depth restriction

$$D \le \frac{L}{10}; \qquad G(7) = \left(\frac{10D}{L}\right) - 1 \le 0$$

2.3. Objective function

The index which is well-defined criterion by which optimality is judged for different combinations of design variables is called the objective function or merit function. Performance level, cost or any other specific quality which can be meaningfully quantified in terms of design variables is acceptable as objective function. Usually the most important design property is selected as the objective function.

For optimization of rectangular beams total cost which includes costs of concrete, labour, steel and shuttering has been taken as the objective function to be minimized. The local prevailing rates in the market have been considered for analysis. The objective function and constraints are represented as the functions of width and depth, which are taken as design variables. For optimization of trapezoidal beams total cost which includes costs of concrete, labour, steel and shuttering has been taken as the objective function to be minimized.

2.3.1. Objective function formulation for rectangular beams

The objective function in terms of the design variables is arrived as under.

Total compressive force $C = 0.4\, f_{ck}\, x_u\, b$

Total tensile force $T = 0.87\, f_y\, A_{st}$

On equating C and T, $\frac{X_u}{d} = \frac{2.175 f_y A_{st}}{f_{ck} bd}$

For under-reinforced section

M_u = Force × Lever arm

$$M_u = 0.87 f_y A_{st}(d - 0.5X_u) = 0.87 f_y A_{st} d\left[1 - \frac{1.088 f_y A_{st}}{f_{ck} bd}\right]$$

On simplification,

$$A_{st}^2 - \frac{0.92 f_y bd}{f_y} A_{st} + \frac{1.055 f_{ck} M_u b}{f_y^2} = 0$$

$$A_{st} = \frac{0.46 f_{ck} bd}{f_y} - \sqrt{\left(\frac{0.212 f_{ck}^2 b^2 d^2}{f_y^2} - \frac{1.055 f_{ck} M_u b}{f_y^2}\right)}$$

$$A_{st} = \frac{0.414 f_{ck} bD}{f_y} - \sqrt{\left(\frac{0.17 f_{ck}^2 b^2 D^2}{f_y^2} - \frac{1.055 f_{ck} M_u b}{f_y^2}\right)} \qquad \text{since } D = 1.111\, d$$

Considering unit rates for concrete, steel and shuttering as 2900 Rs/m^3, 30 Rs/kg and 150 Rs/m^2 respectively.

Concrete and Labour costs; OBJ1 = $bD \times (2900) \times 10^{-6}$

Steel cost; OBJ2 = $\frac{0.414 f_{ck} bD}{f_y} - \sqrt{\left(\frac{0.17 f_{ck}^2 b^2 D^2}{f_y^2} - \frac{1.055 f_{ck} M_u b}{f_y^2}\right)} \times (7850 \times 30) \times 10^{-6}$

Shuttering cost; OBJ3 = (b+2D) × (150) × 10^{-6}
Total cost; OBJ = OBJ1 + OBJ2 + OBJ3

2.3.2. Objective function formulation for trapezoidal beams

For trapezoidal beams the objective function in terms of design variables can be formulated as explained here under.

Top width $b = 100 + 2D \tan\alpha$

Width at x_u from the top $b_1 = 100 + 2(D - x_u)\tan\alpha$

Area of compression zone $A_{top} = \left(\frac{b+b_1}{2}\right)x_u = 100\,x_u + 2D\,x_u \tan\alpha - x_u^2 \tan\alpha$

Point of application of compressive force from top

$$\bar{y} = \left(\frac{b+2b_1}{b+b_1}\right)\frac{x_u}{3} = \left(\frac{100+2D\tan\alpha - 1.33x_u\tan\alpha}{200+4D\tan\alpha - 2x_u\tan\alpha}\right)x_u$$

Total compressive force $C = 0.4 f_{ck}(100\,x_u + 2D\,x_u \tan\alpha - x_u^2 \tan\alpha)$

Total tensile force $T = 0.87 f_y A_{st}$

For an under-reinforced section

M_u = Force × Lever arm $0.87 f_y A_{st}(d - \bar{y})$

$$= 0.87 f_y A_{st}\left[\frac{D}{1.111} - \left(\frac{100+2D\tan\alpha - 1.33x_u\tan\alpha}{200+4D\tan\alpha - 2x_u\tan\alpha}\right)x_u\right] \quad \text{since } D = 1.111\,d$$

Also, $M_u = C(d - \bar{y})$

$$M_u = 0.4 f_{ck}(100x_u + 2Dx_u\tan\alpha - x_u^2\tan\alpha)\left[\frac{D}{1.111} - \left(\frac{100+2D\tan\alpha - 1.33x_u\tan\alpha}{200+4D\tan\alpha - 2x_u\tan\alpha}\right)x_u\right]$$

Method of solution:

Assume a value of x_u

$$0.4 f_{ck}(100x_u + 2Dx_u\tan\alpha - x_u^2\tan\alpha) = C$$

$$x_u^2\tan\alpha - (100+2D\tan\alpha)x_u + \frac{C}{0.4 f_{ck}} = C \qquad \text{where } C = \frac{M_u}{\left(\frac{D}{1.111} - \bar{y}\right)}$$

$$x_{unew} = \frac{\left((100+2D\tan\alpha) - \sqrt{(100+2D\tan\alpha)^2 - 4\tan\alpha\frac{C}{0.4 f_{ck}}}\right)}{2\tan\alpha}$$

If absolute value of x_u and x_{unew}<5.0 then adopt the latest value of x_{unew} for calculation or else repeat the procedure by taking $x_u = x_{unew}$

$\bar{y}$ is calculated using x_{unew}

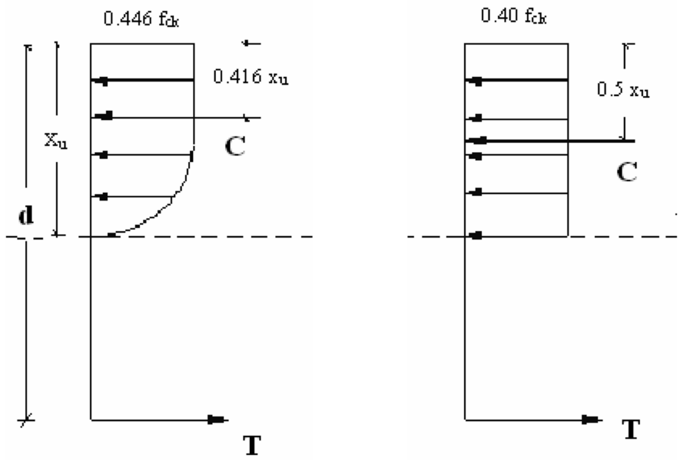

Fig. 3 (a) Rectangular-parabolic stress block (b) Equivalent Rectangular stress block

$$C_1 = C = \frac{M_u}{\left(\frac{D}{1.111} - \bar{y}\right)}$$

$$A_{st} = \frac{C_1}{0.87 f_y}$$

Considering unit rates as presented earlier.
Concrete and Labour costs; OBJ1 = $(100D + D^2 \tan \alpha) \times (2900) \times 10^{-6}$
Steel cost; OBJ2 = $A_{st} \times (7850 \times 30) \times 10^{-6}$
Shuttering cost; OBJ3 = $(100 + 2D\sqrt{1+\tan^2\alpha}) \times (150) \times 10^{-3}$
Total cost; OBJ = OBJ1 + OBJ2 + OBJ3

2.4. Equivalent rectangular stress block (ERSB)

Flexural strength computation of RC sections is possible only if relation between stress and strain is assumed properly. Code of practice gives the liberty to designers to assume any shape of stress block with the condition that strength prediction should be in substantial agreement with test results. The most widely used rectangular-parabolic stress block as recommended by IS-456-2000 (Fig. 3(a)) poses computational problems when odd shaped sections or prismatic sections, where bending associated with direct forces are to be addressed. To overcome the computational rigor associated with rectangular parabolic stress block equivalent rectangular stress block (Fig. 3(b)) is employed which drastically reduces the computational efforts.

2.5. SUMT – An algorithm for shape optimization

Generally in structural optimization the objective function and constraints are non-linear functions of the design variables. The most general view held is that non-linear programming techniques are the most powerful for solving such problems. Also all other techniques can be regarded as special cases of non-linear programming. Sequential Unconstrained Minimization Technique (SUMT) is highly advantageous and best suited for structural optimization. In this technique the problem is converted into an unconstrained one by the introduction of a penalty function and the solution is sought by unconstrained minimization of the transformed function for a sequence of values of penalty parameter. The solution is brought to converge to that of the original constrained problem. Hence this method is also called the penalty function method. The versatility is attributed to the sequential nature of the method that allows gradual or sequential approach to criticality. In addition the technique permits coarser approximations at initial stages and finer at later stages. The method renders itself to minimization of a wide range of functions and is reliable.

The program is written in FORTRAN 77 and can be used for a general non-linear programming problem. The solution procedure is based on the interior penalty function method coupled with Davidon-Fletcher-Powell method of Unconstrained Minimization and Cubic Interpolation method of one dimensional search.

3 Results and discussions

The program run gets terminated either when the specified total number of iterations are completed or when further iterations lead to deterioration of solution which are indicated by messages on the console. The search for optimum should begin with an interior feasible point which is generally known in majority of the problem. It is recommended that high starting feasible values for design variable at the expense of merit function make the approach simple though this approach needs more iterations. Output shows for sequentially reducing values of penalty parameters starting values of design variables get altered and merit function value drops and approaches the minimum and the values of design variable at this stage are the optimum.

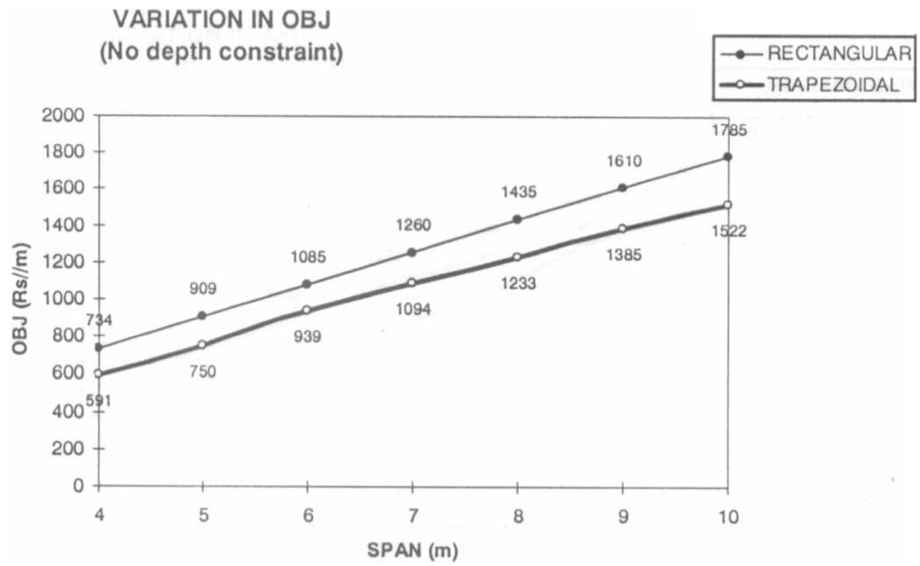

Fig. 4 Objective functions of rectangular and trapezoidal beams

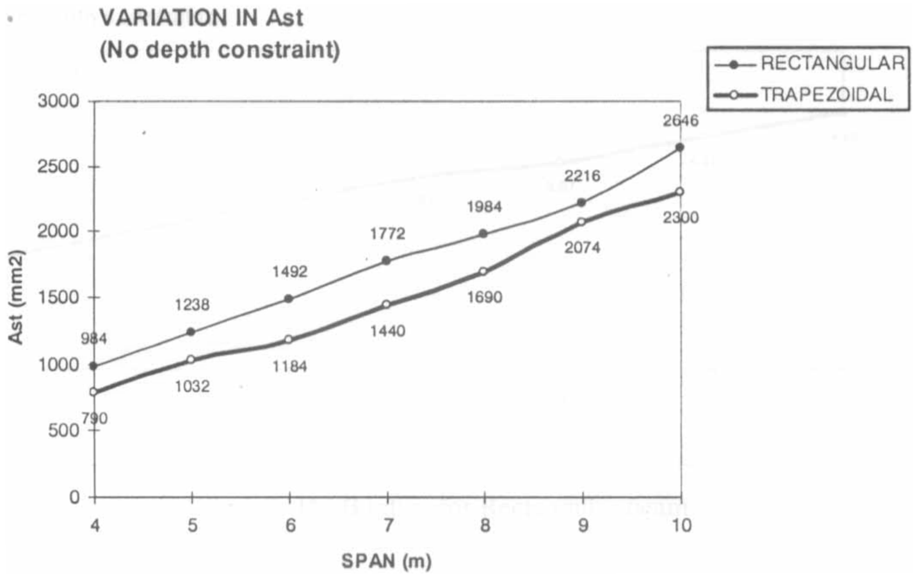

Fig. 5 Areas of steel for rectangular and trapezoidal beams

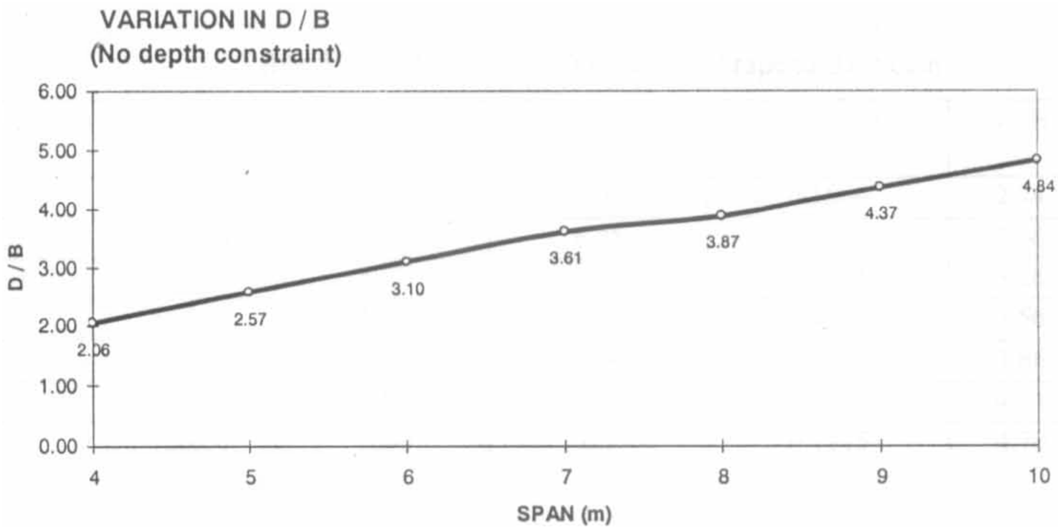

Fig. 6 *D/B* ratios for rectangular beam

Case 1

Here the optimization has been attempted for a simply supported beam of spans ranging from 4m to 10m. Concrete grade M_{20} and Steel grade Fe_{415} have been considered. An all inclusive uniformly distributed load of 60 kN/m (factored) has been taken which is the most usual load encountered in the field.

Fig. 4 presents the variation of merit function with span for both rectangular and trapezoidal beams. From the results presented it is clear that trapezoidal beams are always more economical than their rectangular counterparts and a maximum saving of about 20% in cost is possible if trapezoidal section is chosen in place of rectangle. Trapezoidal beam also consumes less steel than

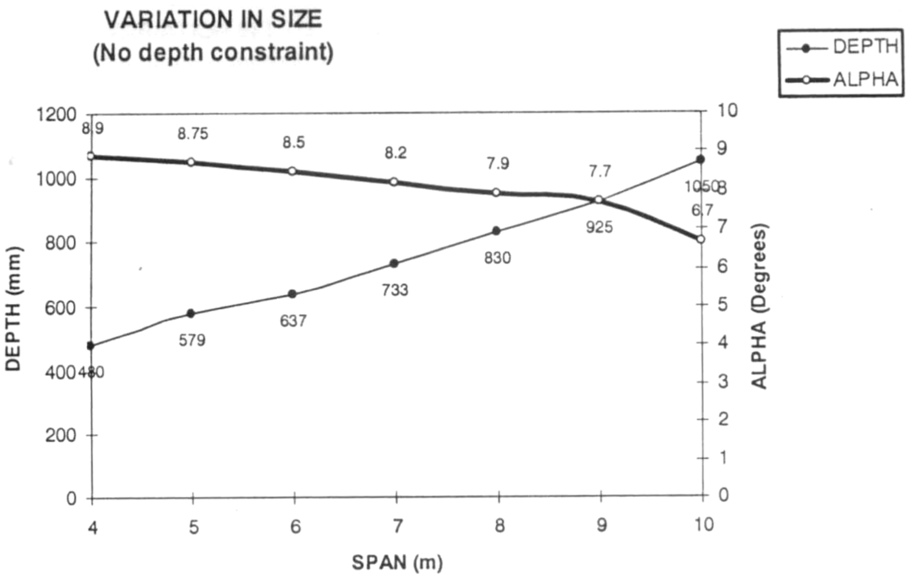

Fig. 7 Shape details for trapezoidal beam

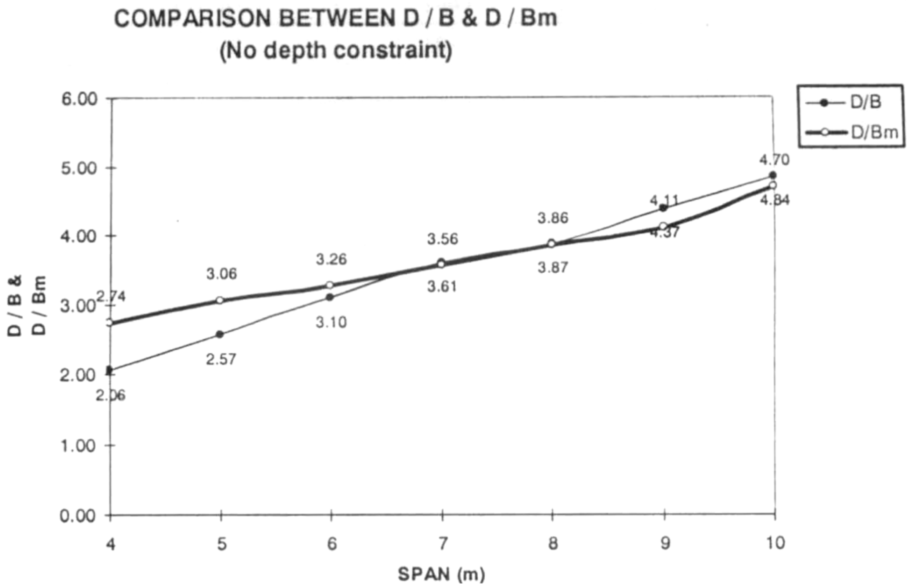

Fig. 8 D/B of rectangular beam and D/B_m of trapezoidal beam

rectangular beam as shown in Fig. 5.

Fig. 6 gives the optimum *D/B* ratios for rectangular beams. It is prudent to note that the optimum D/B ratio increases with increase in span. Size details of trapezoidal beam are presented in Fig. 7. D/B_m ratios increase as span increases and angle α decreases with increase in span.

Fig. 8 shows that D/B_m ratio is more than *D/B* ratio for span less than 7 m and other way round for span more than 7 m indicating shallower beams are possible as span gets longer if trapezoidal beams are adopted. Hence in situations where depths are restricted trapezoidal beams are more

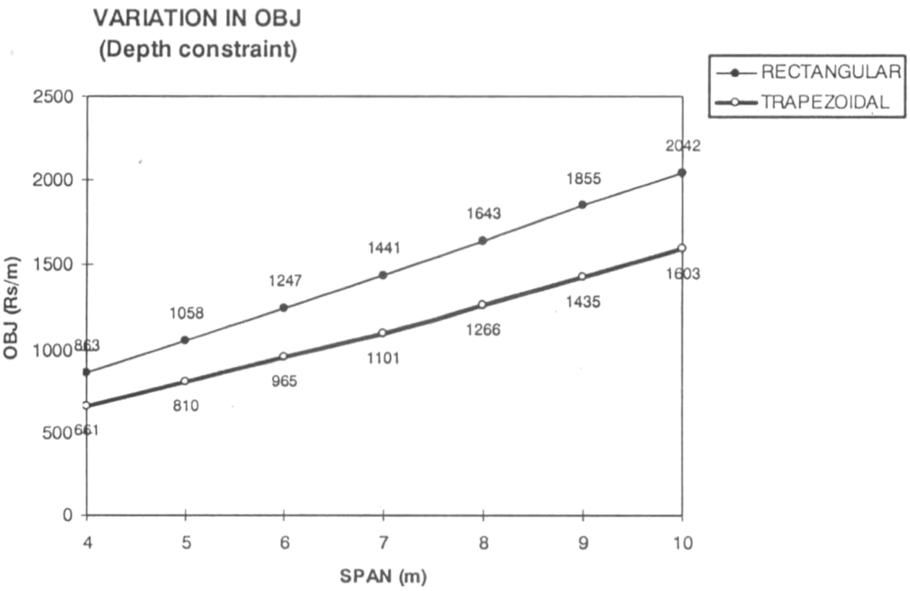

Fig. 9 Objective functions of rectangular and trapezoidal beams

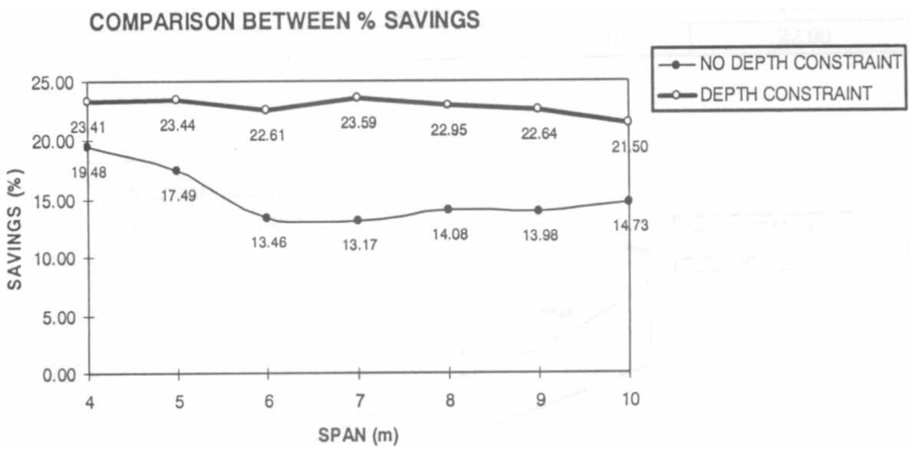

Fig. 10 Percentage savings

meaningful than rectangular beams.

Case 2

Here the optimization has been done for a simply supported beam of spans ranging from 4m to 10m by imposing a constraint that is (D < 1/10th span). Concrete grade M_{20} and Steel grade Fe_{415} have been considered. An all inclusive UDL of 60 kN/m (factored) has been taken which is the most usual load encountered in field. From fig. 6, it is clear that trapezoidal beams are always more economical than their rectangular counterparts.

Fig. 10 shows percentage saving versus span for depth constraint and no depth constraint. Interestingly percentage saving is higher when there is a limitation for depth. The savings in cost

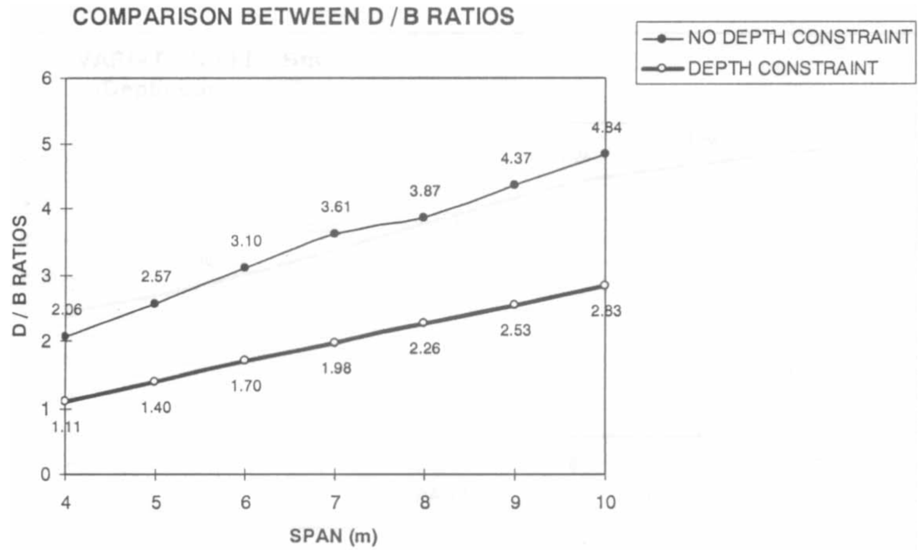

Fig. 11 *D/B* ratios for rectangular beam

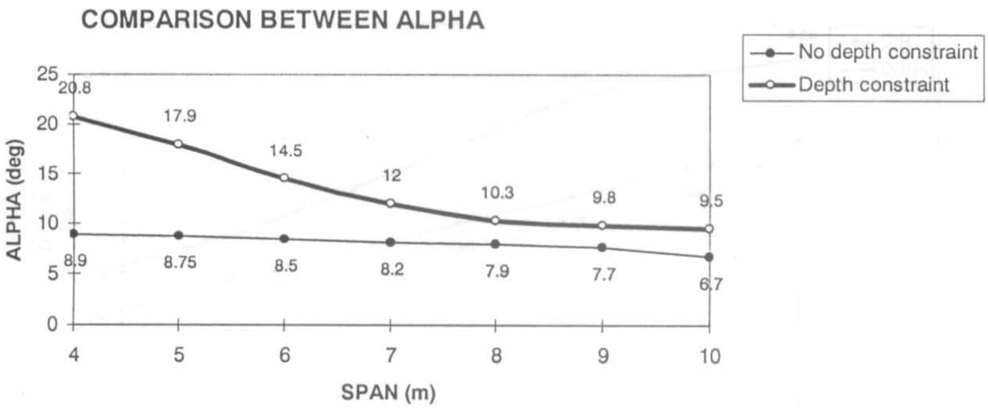

Fig. 12 Variation of alpha with span for no depth constraint and depth constraint cases

(objective function) is more than 20% as observed from the graph.

Fig. 11 shows *D/B* ratios Vs span plotted for a rectangular beam. It is evident that *D/B* ratio for depth constraint case increases linearly with the span. However *D/B* ratio is less than that obtained for case where depth is not restricted.

Fig. 12 shows variation of Alpha Vs span for no depth constraint and depth constraint cases. Though alpha decreases with span in both the cases, its value is higher for the case 2 where the depth is restricted. This case where the depth is restricted indicates that one-way of satisfying the strength and serviceability requirements optimally is to adopt trapezoidal beams.

4 Conclusions

Contemporary architecture and construction techniques are demanding long span beams of the simply supported types in low rise buildings of the shopping mall kind and high rise structures, wherein the tube-in-tube concept to enhance stability and lateral load resistance. Also long span girders are employed in bridges and aqueducts. Any attempt to optimize the overall cost of such structures should begin with search for best conceivable shapes and exploitation of materials strength in case of composites. The work presented is an attempt in this direction, where SUMT has been adopted for shape optimization.

The following conclusions are drawn from the results discussed (i) Results of shape optimization for flexure in beams obtained, clearly indicate that efficacy of trapezoidal section should be seriously reconsidered by the designers. It is obvious that any effort to remove concrete that is ineffective in tension zone and its inclusion in the more effective compression zone is economical. This, perhaps, has not been seriously considered due to decisions that are based on convenience rather than logics (ii) It is prudent here to point out that since the objective function accounts for all parameters affecting the cost, and since trapezoidal beams happen to be less costlier than their rectangular counterparts, their usage should be encouraged (iii) Strength computation of odd shaped section is really cumbersome when rectangular-parabolic stress block is adopted. Tremendous saving in time and effort can be achieved if equivalent rectangular stress block is employed and (iv) SUMT is best suited to address problems of the present kind where merit function and constraints are non-linear functions of the design variables.

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