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Theoretical formulation of double scalar damage variables

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Abstract. The predictive utility of a damage model depends heavily on its particular choice of a damage variable, which serves as a macroscopic approximation in describing the underlying micromechanical processes of microdefects. In the case of spatially perfectly randomly distributed microcracks or microvoids in all directions, isotropic damage model is an appropriate choice, and scalar damage variables were widely used for isotropic or one-dimensional phenomenological damage models. The simplicity of a scalar damage representation is indeed very attractive. However, a scalar damage model is of somewhat limited use in practice. In order to entirely characterize the isotropic damage behaviors of damaged materials in multidimensional space, a system theory of isotropic double scalar damage variables, including the expressions of specific damage energy release rate, the coupled constitutive equations corresponding to damage, the conditions of admissibility for two scalar damage effective tensors within the framework of the thermodynamics of irreversible processes, was provided and analyzed in this study. Compared with the former studies, the theoretical formulations of double scalar damage variables in this study are given in the form of matrix, which has many features such as simpleness, directness, convenience and programmable characteristics. It is worth mentioning that the above-mentioned theoretical formulations are only logically reasonable. Owing to the limitations of time, conditions, funds, etc. they should be subject to multifaceted experiments before their innovative significance can be fully verified. The current level of research can be regarded as an exploratory attempt in this field.

Keywords: double scalar damage variables; damage energy release rate; isotropic; irreversible thermodynamics; theoretical analysis

1. Introduction

Since the introduction of the scalar damage concept by Kachanov (1958) and Rabotnov (1963) for creep of metals, continuum damage mechanics has become an emerging field of active research. Continuum damage mechanics, a combination of the internal state variable theory and the physical considerations of the irreversible thermodynamic processes, provides a powerful framework for derivation of consistent constitutive models suitable for many engineering problems (e.g. Chow and Wei 1999, Zhou *et al*. 2002, Selvadurai 2004, Ohata and Toyoda 2006, Wang *et al*. 2006, Thakkar and Panley 2007, Umit *et al*. 2007, Voyiadjis *et al*. 2008, Voyiadjis *et al*. 2009, Li and Li 2010, Mitsuru *et al*. 2010, Alexandrov and Jeng 2011, Mao *et al*. 2012, Tan and Watanabe 2012, Alexander *et al*. 2012, D'Annibale and Luongo 2013, Pham and Marigo 2013, Misra and Singh 2013, Rinaldi 2013, Xiong *et al*. 2015, Fang *et al*. 2016). The damage variable (or tensor), based on the effective stress concept, represents average material degradation, which reflects the various types of damage at the micro-scale level like nucleation and growth of voids, cracks, cavities, micro-cracks, and other microscopic defects (Voyiadjis and Kattan 2009). The damage variable is scalar in the case isotropic damage mechanics, and the evolution equations are easy to handle (Jarić *et al*. 2012). Lemaitre (1984) argued that it is sufficient for using the assumption of isotropic damage to give good predictions of the load-carrying capacity, the number of cycles, or the time to local failure in structural components. However, some researchers reckon that a single scalar damage variable is not sufficient to entirely characterize isotropic damage behaviors of damaged materials in multidimensional space. For instance, it has been shown by Ju (1990) and Cauvin and Testa (1999) that the accurate and consistent description of the special case of isotropic damage is obtained by using two independent damage variables.

For isotropic damage theory, in which damage is characterized by a single scalar damage variable, the effective Poisson's ratio is immune from the effect of damage, i.e., $v^*=v$, thus, the following result can be obtained $K^*/K=G^*/G=E^*/E$, where K, G, and E are the bulk modulus,the shear modulus and the Young's modulus of undamaged materials, respectively. K^*, G^*, and E^* are their corresponding effective values due to the effect of damage. However, foran isotropic solid material with randomly oriented cracks, the analytical results of micro-mechanics theory (Tang *et al*. 2002) show that $G^*/G>E^*/E$ when the crack density parameter $0<\beta<1$. Therefore, strictly speaking, the isotropic property of material must be characterized by two independent scalar parameters.

In order to entirely characterize the isotropic damage behaviors of damaged materials inmultidimensional space,

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we deeply investigated the isotropic property of damaged materialsand established a system theory of isotropic double scalar damage variables on the basis of the research of Tang *et al*. (2002), including the expressions of specific damage energy release rate, the coupled constitutive equations corresponding to damage, the conditions of admissibility for two scalar damage effective tensors within the framework of the thermodynamics of irreversible processes. Compared with the research of Tang *et al*. (2002), the theoretical formulations of double scalar damage variables in this study are given in the form of matrix, which has many features such as simpleness, directness, convenience and programmable characteristics. In addition, it should be noted that the proposed isotropic double scalar damage model has been successfully applied in analyzing the consolidation problem of damaged porous media (Xue *et al*. 2014). Different from Xue *et al*. (2014), however, the present paper mainly concentrates on introducing and discussing the entire theory of isotropic double scalar damage variables instead of applications.

2. Isotropic double scalar damage variables

Young's modulus (E), Poisson's ratio (v), shear modulus (G or μ) and bulk modulus (K) are the elastic material constants commonly used in engineering. Four corresponding damage parameters, Ω_E, Ω_v, Ω_G, and Ω_k can be traditionally defined in terms of the effective engineering elastic coefficients and to characterize phenomenologically the state of damage as (Xue 2008, Zhang and Cai 2010)

$$\Omega_E = 1 - E^* / E \tag{1a}$$

$$\Omega_v = 1 - v^* / v \tag{1b}$$

$$\Omega_G = 1 - G^* / G \tag{1c}$$

$$\Omega_K = 1 - K^* / K \tag{1d}$$

As an example, taking into account the elastic relations of both the undamaged and the damaged material

$$\lambda = \frac{G(E-2G)}{3G-E} \quad \text{and} \quad \lambda^* = \frac{G^*(E^*-2G^*)}{3G^*-E^*} \tag{2}$$

where λ is the Lamé constant. λ^* is the effective Lamé constant for isotropic damaged material. By taking the following classical elastic relation into account

$$v = \frac{E}{2G} - 1, \quad K = \frac{GE}{3(3G-E)} \tag{3}$$

A similar relationship for isotropic damaged material can be derived, such as

$$v^* = \frac{E(1-\Omega_E)}{2G(1-\Omega_G)} - 1, \quad K^* = \frac{GE(1-\Omega_G)(1-\Omega_E)}{3[3G(1-\Omega_G) - E(1-\Omega_E)]} \tag{4}$$

The isotropic damage behavior can be characterized by two scalar damage variables. Any two out of the four damage parameters Ω_E, Ω_v, Ω_G, and Ω_k can be used to quantify isotropic damage. After selection of the two independent damage variables, the corresponding formulas estimating the two derived damage parameters can be obtained (Xue 2008, Zhang and Cai 2010).

3. Strain energy release rate with double scalar damage variables

The principle of energy equivalence has been proposed in many articles, the elastic energy for a damaged material is the same as that of the undamaged material when the stress tensor is replaced by the corresponding effective stress in the stress-based form. Mathematically, it is shown again (Xue 2008, Zhang and Cai 2010)

$$W^*(\{\sigma\}, \Omega_E, \Omega_G) = \frac{1}{2}\{\sigma\}^{\mathrm{T}} [D^*]^{-1} \{\sigma\} \tag{5}$$

where W^* stands for the elastic energy, the effective elastic compliance tensor $[D^*]^{-1}$ in terms of the two scalar damage variables Ω_E and Ω_G for the thermodynamics system can be expressed as (Xue 2008)

$$[D^*]^{-1} = \left[\begin{array}{ccc|ccc}
\frac{1}{E(1-\Omega_E)} & \frac{1}{E(1-\Omega_E)} - \frac{1}{2\mu(1-\Omega_\mu)} & \frac{1}{E(1-\Omega_E)} - \frac{1}{2\mu(1-\Omega_\mu)} & & & \\
\frac{1}{E(1-\Omega_E)} - \frac{1}{2\mu(1-\Omega_\mu)} & \frac{1}{E(1-\Omega_E)} & \frac{1}{E(1-\Omega_E)} - \frac{1}{2\mu(1-\Omega_\mu)} & & 0 & \\
\frac{1}{E(1-\Omega_E)} - \frac{1}{2\mu(1-\Omega_\mu)} & \frac{1}{E(1-\Omega_E)} - \frac{1}{2\mu(1-\Omega_\mu)} & \frac{1}{E(1-\Omega_E)} & & & \\
\hline
 & & & \frac{1}{\mu(1-\Omega_\mu)} & 0 & 0 \\
 & 0 & & 0 & \frac{1}{\mu(1-\Omega_\mu)} & 0 \\
 & & & 0 & 0 & \frac{1}{\mu(1-\Omega_\mu)}
\end{array}\right] \tag{6}$$

By substituting the above equation into Eq. (5), it gives

$$W^* = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}^{\mathrm{T}} \left[\begin{array}{ccc|ccc}
\frac{1}{E(1-\Omega_E)} & \frac{1}{E(1-\Omega_E)} - \frac{1}{2\mu(1-\Omega_\mu)} & \frac{1}{E(1-\Omega_E)} - \frac{1}{2\mu(1-\Omega_\mu)} & & & \\
\frac{1}{E(1-\Omega_E)} - \frac{1}{2\mu(1-\Omega_\mu)} & \frac{1}{E(1-\Omega_E)} & \frac{1}{E(1-\Omega_E)} - \frac{1}{2\mu(1-\Omega_\mu)} & & 0 & \\
\frac{1}{E(1-\Omega_E)} - \frac{1}{2\mu(1-\Omega_\mu)} & \frac{1}{E(1-\Omega_E)} - \frac{1}{2\mu(1-\Omega_\mu)} & \frac{1}{E(1-\Omega_E)} & & & \\
\hline
 & & & \frac{1}{\mu(1-\Omega_\mu)} & 0 & 0 \\
 & 0 & & 0 & \frac{1}{\mu(1-\Omega_\mu)} & 0 \\
 & & & 0 & 0 & \frac{1}{\mu(1-\Omega_\mu)}
\end{array}\right] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix} \tag{7}$$

Based on the laws of thermodynamics with internal state variables, the damage strain energy release rates corresponding to the damage variables Ω_E and Ω_μ can be defined as

$$Y_E = \frac{\partial W^*}{\partial \Omega_E} \tag{8}$$

$$Y_\mu = \frac{\partial W^*}{\partial \Omega_\mu} \tag{9}$$

Substituting Eq. (7) into Eqs. (8) and (9), it gives

$$Y_E = \frac{1}{2}\left\{\begin{matrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{matrix}\right\}^{\mathrm{T}}\left[\begin{array}{ccc|ccc}\frac{1}{E\left(1-\Omega_E\right)^2} & \frac{1}{E\left(1-\Omega_E\right)^2} & \frac{1}{E\left(1-\Omega_E\right)^2} & & & \\ \frac{1}{E\left(1-\Omega_E\right)^2} & \frac{1}{E\left(1-\Omega_E\right)^2} & \frac{1}{E\left(1-\Omega_E\right)} & & 0 & \\ \frac{1}{E\left(1-\Omega_E\right)^2} & \frac{1}{E\left(1-\Omega_E\right)^2} & \frac{1}{E\left(1-\Omega_E\right)^2} & & & \\ \hline & & & 0 & 0 & 0\\ & 0 & & 0 & 0 & 0\\ & & & 0 & 0 & 0\end{array}\right]\left\{\begin{matrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{matrix}\right\} \tag{10}$$

$$Y_\mu = \frac{1}{2}\left\{\begin{matrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{matrix}\right\}^{\mathrm{T}}\left[\begin{array}{ccc|ccc}0 & \frac{-1}{2\mu\left(1-\Omega_\mu\right)^2} & \frac{-1}{2\mu\left(1-\Omega_\mu\right)^2} & & & \\ \frac{-1}{2\mu\left(1-\Omega_\mu\right)^2} & 0 & \frac{-1}{2\mu\left(1-\Omega_\mu\right)^2} & & 0 & \\ \frac{-1}{2\mu\left(1-\Omega_\mu\right)^2} & \frac{-1}{2\mu\left(1-\Omega_\mu\right)^2} & 0 & & & \\ \hline & & & \frac{1}{\mu\left(1-\Omega_\mu\right)^2} & 0 & 0\\ & 0 & & 0 & \frac{1}{\mu\left(1-\Omega_\mu\right)^2} & 0\\ & & & 0 & 0 & \frac{1}{\mu\left(1-\Omega_\mu\right)^2}\end{array}\right]\left\{\begin{matrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{matrix}\right\} \tag{11}$$

The overall specific damage energy release rate should be the sum of Y_E and Y_μ i.e.,

$$Y = Y_E + Y_\mu$$

$$= \frac{1}{2}\left\{\begin{matrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{matrix}\right\}^{\mathrm{T}}\left[\begin{array}{ccc|ccc}\frac{1}{E\left(1-\Omega_E\right)^2} & \frac{1}{E\left(1-\Omega_E\right)^2}-\frac{1}{2\mu\left(1-\Omega_\mu\right)^2} & \frac{1}{E\left(1-\Omega_E\right)^2}-\frac{1}{2\mu\left(1-\Omega_\mu\right)^2} & & & \\ \frac{1}{E\left(1-\Omega_E\right)^2}-\frac{1}{2\mu\left(1-\Omega_\mu\right)^2} & \frac{1}{E\left(1-\Omega_E\right)^2} & \frac{1}{E\left(1-\Omega_E\right)^2}-\frac{1}{2\mu\left(1-\Omega_\mu\right)^2} & & 0 & \\ \frac{1}{E\left(1-\Omega_E\right)^2}-\frac{1}{2\mu\left(1-\Omega_\mu\right)^2} & \frac{1}{E\left(1-\Omega_E\right)^2}-\frac{1}{2\mu\left(1-\Omega_\mu\right)^2} & \frac{1}{E\left(1-\Omega_E\right)^2} & & & \\ \hline & & & \frac{1}{\mu\left(1-\Omega_\mu\right)^2} & 0 & 0\\ & 0 & & 0 & \frac{1}{\mu\left(1-\Omega_\mu\right)^2} & 0\\ & & & 0 & 0 & \frac{1}{\mu\left(1-\Omega_\mu\right)^2}\end{array}\right]\left\{\begin{matrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{matrix}\right\} \tag{12}$$

It is known that the specific elastic energy W can be considered as a sum of two parts

$$W^* = W_b + W_d \tag{13}$$

The first part W_b reflects the energy due to bulk change (i.e., hydrostatic energy) while the second one W_d is the contribution due to distortion of damaged material (i.e., shear energy). It is obvious that

$$W_b = \left[\frac{1}{2E\left(1-\Omega_E\right)} - \frac{1}{6\mu\left(1-\Omega_\mu\right)}\right]\left\{\begin{matrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{matrix}\right\}^{\mathrm{T}}\left[\begin{array}{ccc|ccc}1 & 1 & 1 & & & \\ 1 & 1 & 1 & & 0 & \\ 1 & 1 & 1 & & & \\ \hline & & & 0 & 0 & 0\\ & 0 & & 0 & 0 & 0\\ & & & 0 & 0 & 0\end{array}\right]\left\{\begin{matrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{matrix}\right\} \tag{14}$$

$$W_d = \frac{1}{4\mu\left(1-\Omega_\mu\right)}\left\{\begin{matrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{matrix}\right\}^{\mathrm{T}}\left[\begin{array}{ccc|ccc}\frac{2}{3} & 0 & 0 & & & \\ 0 & \frac{2}{3} & 0 & & 0 & \\ 0 & 0 & \frac{2}{3} & & & \\ \hline & & & 2 & 0 & 0\\ & 0 & & 0 & 2 & 0\\ & & & 0 & 0 & 2\end{array}\right]\left\{\begin{matrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{matrix}\right\} \tag{15}$$

According to Eq. (13), we have

$$\begin{aligned} W^* &= W_b + W_d \\ &= \left[\frac{1}{2E\left(1-\Omega_E\right)} - \frac{1}{6\mu\left(1-\Omega_\mu\right)}\right]\left\{\begin{matrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{matrix}\right\}^{\mathrm{T}}\left[\begin{array}{ccc|ccc}1 & 1 & 1 & & & \\ 1 & 1 & 1 & & 0 & \\ 1 & 1 & 1 & & & \\ \hline & & & 0 & 0 & 0\\ & 0 & & 0 & 0 & 0\\ & & & 0 & 0 & 0\end{array}\right]\left\{\begin{matrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{matrix}\right\} \\ &\quad + \frac{1}{4\mu\left(1-\Omega_\mu\right)}\left\{\begin{matrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{matrix}\right\}^{\mathrm{T}}\left[\begin{array}{ccc|ccc}\frac{2}{3} & 0 & 0 & & & \\ 0 & \frac{2}{3} & 0 & & 0 & \\ 0 & 0 & \frac{2}{3} & & & \\ \hline & & & 2 & 0 & 0\\ & 0 & & 0 & 2 & 0\\ & & & 0 & 0 & 2\end{array}\right]\left\{\begin{matrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{matrix}\right\} \end{aligned} \tag{16}$$

Lets the average stress be

$$\sigma_m = \left(\sigma_x + \sigma_y + \sigma_z\right)/3 \text{ or } \sigma_m = \sigma_{kk}/3 \tag{17}$$

and the von Mises equivalent stress be

$$\sigma_{eq} = \left[\frac{2}{3}\{s\}^{\mathrm{T}}\{s\}\right]^{\frac{1}{2}} \quad \text{or} \quad \sigma_{eq} = \left(\frac{2}{3}s_{ij}s_{ij}\right)^{\frac{1}{2}} \tag{18a}$$

$$\sigma_{eq} = \left\{\frac{2}{3}\left[\left(\sigma_x-\sigma_m\right)^2 + \left(\sigma_y-\sigma_m\right)^2 + \left(\sigma_z-\sigma_m\right)^2 + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right]\right\}^{\frac{1}{2}} \tag{18b}$$

where s_{ij} is the deviatoric components of stress tensor, i.e.,

$$\left\{s_{ij}\right\} = \left\{\sigma_x-\sigma_m, \sigma_y-\sigma_m, \sigma_z-\sigma_m, \tau_{xy}, \tau_{yz}, \tau_{zx}\right\}^{\mathrm{T}} \tag{19a}$$

$$s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3 \tag{19b}$$

Taking Eqs. (18) and (19) into account, Eqs. (14)-(16) can be rewritten

$$W_b=\frac{1}{2}\left[\frac{9}{E\left(1-\Omega_E\right)}-\frac{3}{\mu\left(1-\Omega_\mu\right)}\right]\sigma_m^2 \tag{20a}$$

and

$$W_d=\frac{1}{6\mu\left(1-\Omega_\mu\right)}\sigma_{eq}^2 \tag{20b}$$

$$W^*=W_b+W_d=\frac{1}{2}\left[\frac{9}{E\left(1-\Omega_E\right)}-\frac{3}{\mu\left(1-\Omega_\mu\right)}\right]\sigma_m^2+\frac{1}{6\mu\left(1-\Omega_\mu\right)}\sigma_{eq}^2 \tag{20c}$$

Substituting Eq. (20) into Eqs. (8) and (9), the specific damage strain energy release rate has four terms

$$Y_E^b=\frac{\partial W_b}{\partial \Omega_E}=\frac{9\sigma_m^2}{2E\left(1-\Omega_E\right)^2} \tag{21a}$$

$$Y_\mu^b=\frac{\partial W_b}{\partial \Omega_\mu}=-\frac{3\sigma_m^2}{2\mu\left(1-\Omega_\mu\right)^2} \tag{21b}$$

$$Y_E^d=\frac{\partial W_d}{\partial \Omega_E}=0 \tag{21c}$$

$$Y_\mu^d=\frac{\partial W_d}{\partial \Omega_\mu}=\frac{\sigma_{eq}^2}{6\mu\left(1-\Omega_\mu\right)^2} \tag{21d}$$

The specific damage strain energy release rate corresponding to the change of both bulk and distortion are respectively

$$Y^b=Y_E^b+Y_\mu^b=\frac{9\sigma_m^2}{2E\left(1-\Omega_E\right)^2}-\frac{3\sigma_m^2}{2\mu\left(1-\Omega_\mu\right)^2} \tag{22a}$$

$$Y^d=Y_E^d+Y_\mu^d=\frac{\sigma_{eq}^2}{6\mu\left(1-\Omega_\mu\right)^2} \tag{22b}$$

The alternant expression of isotropic damage strain energy release rate with respect to the double scale damage variables Ω_E and Ω_μ can be expressed as

$$\begin{aligned} Y_E&=\frac{\partial W^*}{\partial \Omega_E}=\frac{9\sigma_m^2}{2E\left(1-\Omega_E\right)^2}, \\ Y_\mu&=\frac{\partial W^*}{\partial \Omega_\mu}=\frac{3\sigma_{eq}^2}{2\mu\left(1-\Omega_\mu\right)^2}\left[\left(\frac{1}{3}\right)^2-\left(\frac{\sigma_m}{\sigma_{eq}}\right)^2\right] \end{aligned} \tag{22c}$$

The total specific damage energy release rate $Y=Y_b+Y_d$, which considers the effect of tensile damage, shear damage and stress triaxiality, may be expressed as

$$Y=\frac{\sigma_{eq}^2}{2E}\left\{\frac{E/\mu}{3\left(1-\Omega_\mu\right)^2}+3\left[\frac{3}{\left(1-\Omega_E\right)^2}-\frac{E/\mu}{\left(1-\Omega_\mu\right)^2}\right]\left(\frac{\sigma_m}{\sigma_{eq}}\right)^2\right\} \tag{23}$$

By taking $E/\mu=2(1+\nu)$, thus

$$Y=\frac{\sigma_{eq}^2}{2E}\left\{\frac{2(1+\nu)}{3\left(1-\Omega_\mu\right)^2}+3\left[\frac{3}{\left(1-\Omega_E\right)^2}-\frac{2(1+\nu)}{\left(1-\Omega_\mu\right)^2}\right]\left(\frac{\sigma_m}{\sigma_{eq}}\right)^2\right\} \tag{24}$$

It is noticed that if $\Omega_E=\Omega_\mu=\Omega$, the above equation can be degenerated the single scalar damage as

$$Y=\frac{\sigma_{eq}^2}{2E\left(1-\Omega\right)^2}\left[\frac{2}{3}\left(1+\nu\right)+3\left(1-2\nu\right)\left(\frac{\sigma_m}{\sigma_{eq}}\right)^2\right] \tag{25}$$

which is the same as the Equation that was firstly presented by Lemaitre (1985). The specific damage energy release rate (Eq. (23) or Eq. (24)) derived from the present model of isotropic damage mechanics with double scalar damage variables is more practical than that of the single scalar damage variable (Eq. (25)) to quantify effects of damage. It should be noted that the damage evolution equations are not presented for the reason that the dissipation potential is not explicitly defined in this study, and readers are referred to Xue *et al*. (2014) concerning with the inelastic dissipation potential function and evolution equations of inelastic damage variables.

4. Coupled constitutive equations corresponding to damage

The total strain is defined as (Xue 2008, Zhang and Cai 2010)

$$\{\varepsilon\}=\{\varepsilon^e\}+\{\varepsilon^{in}\} \tag{26}$$

where $\{\varepsilon^e\}$ the elastic strain, $\{\varepsilon^{in}\}$ the inelastic strain. The damage coupled elastic constitutive equation is written by $\{\varepsilon^e\}=[D^*]^{-1}\{\sigma\}$ or $\{\sigma\}=[D^*]\{\varepsilon^e\}$, where $[D^*]$ is the effective elastic matrix for damaged material defined before.

For derivation of damage-coupled inelastic constitutive equation, the deformation part of the dissipation potential is proposed for the eutectic material as

$$\Phi_{in}^*\left(\{\sigma\},\sigma_m\right)=J_2 \tag{27}$$

where J_2 is the second invariant of the stress deviatoric defined as

$$\Phi_{in}^*\left(\{\sigma\},\sigma_m\right)=J_2=\left\{\frac{3}{2}\{s\}^{\mathrm{T}}\{s\}\right\}^{\frac{1}{2}} \tag{28}$$

Therefore, the inelastic strain rate can be derived as

$$\{\dot{\varepsilon}\}=\lambda_{\mathrm{in}}\frac{\partial \Phi^*}{\partial\{\sigma\}}=\lambda_{\mathrm{in}}\frac{\partial \Phi_{in}^*}{\partial\{\sigma\}}=\lambda_{\mathrm{in}}\frac{3}{2}\frac{(s)}{J_2} \tag{29}$$

From Eqs. (28) and (29), the relationship between the multiplier λ_{in} and the inelastic strain $\{\varepsilon^{in}\}$ can be derived as follows

$$\left\{\dot{\varepsilon}^{in}\right\}^{\mathrm{T}}\left\{\dot{\varepsilon}^{in}\right\}=\lambda_{in}^{2}\left(\frac{3}{2}\right)^{2}\frac{\{s\}^{\mathrm{T}}\{s\}}{J_{2}^{2}}=\frac{3}{2}\lambda_{in}^{2},\quad \lambda_{in}=\left\{\left\{\frac{2}{3}\right\}\left\{\varepsilon^{in}\right\}^{\mathrm{T}}\left\{\varepsilon^{in}\right\}\right\}^{\frac{1}{2}} \tag{30}$$

Therefore, the multiplier λ_{in} is equal to the equivalent inelastic strain rate.

5. Conditions of admissibility for two scalar damage effective tensors

In order to discuss the conditions of admissibility for two scalar damage effective tensors, one of simple form can be presented as follows (Xue 2008, Zhang and Cai 2010)

$$W_{e}^{*}=G^{*}\varepsilon^{2}+\frac{1}{2}\lambda^{*}\left(Tr\varepsilon\right)^{2}=G^{*}e^{2}+\frac{1}{2}K^{*}\left(Tr\varepsilon\right)^{2}>0 \tag{31}$$

where W_e^* is the elastic energy, which requires that the following necessary $G>0$, $\lambda^*>0$, and sufficient $G^*>0$, $K^*>0$ conditions, or the effective Young's modulus $E^*>0$ and the effective Poisson's ratio $-1<\nu^*<0.5$, that is

$$G^{*}=\frac{E^{*}}{2\left(1+\nu^{*}\right)}=\frac{\left(1-\Omega\right)^{2}}{\left(1-\omega\right)^{2}}\frac{E}{2\left(1+\nu\right)}>0 \tag{32a}$$

$$K^{*}=\frac{E^{*}}{2\left(1+\nu^{*}\right)}=\frac{\left(1-\Omega\right)^{2}}{\left(1+2\omega\right)^{2}}\frac{E}{3\left(1-2\nu\right)}>0 \tag{32b}$$

$$-\frac{\nu^{*}}{E}=\frac{\left(1-\Omega\right)^{2}\nu}{E}<0 \tag{32c}$$

where Ω and ω are damage variables.

Nevertheless, the set of the two above conditions must be enriched by the additional condition $\nu^*>0$ since there does not exist materials of negative Poisson's ratio. This leads to the following requirement imposed on elements in $[D^*]^{-1}$, that is

$$-\frac{\nu^{*}}{E}=\frac{\nu-2\left(1-\nu\right)w-\left(1-3\nu\right)w^{2}}{E\left(1-\Omega\right)^{2}}<0 \tag{33}$$

As long as the denominator of fracture Eq. (33) is always positive the numerator changes sign for real materials for which the initial (undamaged) Poisson's ratio is $0<\nu<0.5$. For instance, in the case of solder material 63Sn-37Pb considered by Chow and Wei (1999) for which $\nu=0.4$, $D_{1122}^{-1\,*}$ depends on w as shown in Fig. 1.

The change of sign accompanying $w=0.354$ means that for a damage more advanced than this value the material starts to behave in a peculiar way, namely there is observed elongation in the direction transverse to the direction of the uniaxial tension. This means that $w_{max}=0.354$ is the upper bound of the damage variable w. Consequently, a physically impossible behavior mentioned above was not observed by Chow and Wei (1999). Since the numerical examples presented in the model restrict themselves to uniaxial tension tests and accompanied magnitude of damage variable w is smaller than its theoretical limit. In order to derive conditions of admissibility for the damage effective tensors of double scalar damage variables, let us consider a general fourth rank symmetric tensor similar to the damage effective tensor presented as Eq. (34)

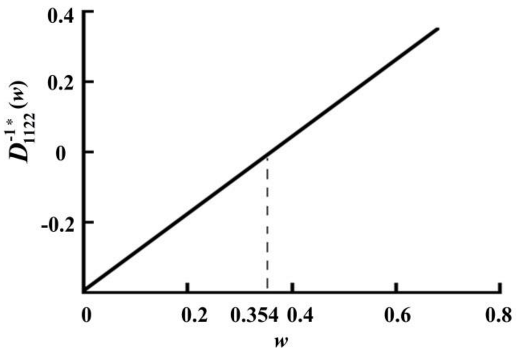

Fig. 1 Dependence of damage modified component $D_{1122}^{-1\,*}$ of constitutive tensor with respect to damage variable w for material 63Sn–37Pb solder

$$[\Psi]=\left[\begin{array}{ccc|ccc} a+b & a & a & & & \\ a & a+b & a & & 0 & \\ a & a & a+b & & & \\ \hline & & & b & 0 & 0 \\ & 0 & & 0 & b & 0 \\ & & & 0 & 0 & b \end{array}\right] \tag{34}$$

The symmetric constitutive Hookean tensor of isotropic material can be expressed as

$$[D]^{-1}=\left[\begin{array}{ccc|ccc} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & 0 & \\ \lambda & \lambda & \lambda+2\mu & & & \\ \hline & & & \mu/2 & 0 & 0 \\ & 0 & & 0 & \mu/2 & 0 \\ & & & 0 & 0 & \mu/2 \end{array}\right] \tag{35}$$

where $a=\lambda$, $b=\mu/2$. In such a case the damage-modified tensor of elasticity remains symmetric and isotropic tensor of four rank can be presented as follows

$$\begin{gathered}\left[\boldsymbol{D}^{*}\right]^{-1}=\left[\underline{\boldsymbol{\Psi}}\right]^{\mathrm{T}}\left[\boldsymbol{D}\right]^{-1}\left[\underline{\boldsymbol{\Psi}}\right]\\ D_{ijkl}^{*\,-1}=\underline{\boldsymbol{\Psi}}_{ijmn}D_{mnpq}^{-1}\underline{\boldsymbol{\Psi}}_{pqkl}=\lambda^{*}\delta_{ij}\delta_{kl}+\mu^{*}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)\end{gathered} \tag{36}$$

The effective Lamé constant λ^* and μ^* can be similarly defined as

$$\lambda^{*}=\left(3a\lambda+2aG+b\lambda\right)\left(3a+b\right)+2abG \tag{37a}$$

$$\mu^{*}=G^{*}=b^{2}G \tag{37b}$$

Necessary and sufficient conditions of positive definiteness of quadratic form associated with the strain energy Eq. (31) enriched by additional condition of Eqs. (32), (33) are as follows

$$K^* = (\lambda + 2G/3)(9a^2 + 6ab + b^2) > 0, \quad G^* = b^2 G$$
$$-\frac{\nu^*}{E^*} = \frac{1}{9K^*} - \frac{1}{6G} = \frac{(1+\nu)(9a^2 + 6ab) + 3\nu b^2}{E(9a^2 + 6ab + b^2)b^2} < 0 \tag{38}$$

It is visible that all these inequalities are satisfied if $a \geq 0$, $b > 0$. The following three cases can be distinguished

Case 1: If $a=0$ and $b=1/(1-\Omega)$, then inequalities

$$K^* = \frac{E^*}{3(1-2\nu^*)} = \frac{E}{3(1-2\nu)(1-\Omega)^2} > 0$$
$$G^* = \frac{E}{2(1+\nu)(1-\Omega)^2} > 0 \tag{39}$$
$$-\frac{\nu^*}{E^*} = -\frac{(1-\Omega)^2 \nu}{E} < 0$$

are always satisfied, so the damage effective tensor of the form

$$[\underline{\Psi}] = [I]/(1-\Omega)$$
$$\text{or} \quad \underline{\Psi}_{ijkl} = (\delta_{ik}\delta_{jk} + \delta_{il}\delta_{jk})/(1-\Omega)/2 \tag{40}$$

is always admissible.

Case 2: If $w=1/(1-\Omega)$ and $b=1/(1-\Omega)$, then inequalities

$$K^* = \frac{9\omega^2 + 6\omega + 1}{(1-\Omega)^2} \frac{E}{3(1-2\nu)} > 0$$
$$G^* = \frac{1}{(1-\Omega)^2} \frac{E}{2(1+\nu)} \tag{41}$$
$$-\frac{\nu^*}{E^*} = \frac{(1-\Omega)^2 \left[(1+\nu)(9\omega^2 + 6\omega) + 3\nu\right]}{3E(9\omega^2 + 6\omega + 1)} < 0$$

are always satisfied so the damage effective tensor being sum of two first terms

$$[\Psi] = \left[\begin{array}{ccc|ccc} \frac{1+\omega}{1-\Omega} & \frac{\omega}{1-\Omega} & \frac{\omega}{1-\Omega} & & & \\ \frac{\omega}{1-\Omega} & \frac{1+\omega}{1-\Omega} & \frac{\omega}{1-\Omega} & & 0 & \\ \frac{\omega}{1-\Omega} & \frac{\omega}{1-\Omega} & \frac{1+\omega}{1-\Omega} & & & \\ \hline & & & \frac{1}{1-\Omega} & 0 & 0 \\ & 0 & & 0 & \frac{1}{1-\Omega} & 0 \\ & & & 0 & 0 & \frac{1}{1-\Omega} \end{array}\right] \tag{42}$$

is always admissible.

Case 3: If $a=1/(1-\Omega)$ and $b=w/(1-\Omega)$ then inequalities

$$K^* = \frac{9 + 6\omega + \omega^2}{(1-\Omega)^2} \frac{E}{3(1-2\nu)} > 0$$
$$G^* = \frac{\omega^2}{(1-\Omega)^2} \frac{E}{2(1+\nu)} \tag{43}$$
$$\frac{\nu^*}{E^*} = -\frac{(1-\Omega)^2 \left[(1+\nu)(9+6\omega) + 3\nu\omega^2\right]}{3E(9 + 6\omega + \omega^2)} < 0$$

are always satisfied so the damage effect tensor

$$[\underline{\Psi}] = \left[\begin{array}{ccc|ccc} \frac{1+\omega}{1-\Omega} & \frac{1}{1-\Omega} & \frac{1}{1-\Omega} & & & \\ \frac{1}{1-\Omega} & \frac{1+\omega}{1-\Omega} & \frac{1}{1-\Omega} & & 0 & \\ \frac{1}{1-\Omega} & \frac{1}{1-\Omega} & \frac{1+\omega}{1-\Omega} & & & \\ \hline & & & \frac{\omega}{1-\Omega} & 0 & 0 \\ & 0 & & 0 & \frac{\omega}{1-\Omega} & 0 \\ & & & 0 & 0 & \frac{\omega}{1-\Omega} \end{array}\right] \tag{44}$$

is always admissible.

Case 4: If $a=[1/(1-\Omega_K)-1/(1-\Omega_u)]/3$ and $b=1/(1-\Omega_\mu)$, then inequalities

$$K^* = \frac{K}{(1-\Omega_K)^2} = \frac{1}{(1-\Omega_K)^2} \frac{E}{3(1-2\nu)} > 0 \tag{45a}$$

$$G^* = \mu^* = \frac{\mu}{(1-\Omega_\mu)^2} = \frac{1}{(1-\Omega_\mu)^2} \frac{E}{2(1+\nu)} > 0 \tag{45b}$$

are always satisfied, however the last condition

$$-\frac{\nu^*}{E^*} = -\frac{(1+\nu)(1-\Omega_\mu)^2 - (1-2\nu)(1-\Omega_K)^2}{3E} < 0 \tag{45c}$$

is not always fulfilled since the numerator may change the sign. Therefore the damage effective tensor referring to damage variables affected separately the volumetric and deviatoric parts of the stress can be presented as follows

$$[\underline{\Psi}] = \frac{1}{3}\left[\begin{array}{ccc|ccc} \frac{1}{1-\Omega_K} + \frac{2}{1-\Omega_\mu} & \frac{1}{1-\Omega_K} - \frac{1}{1-\Omega_\mu} & \frac{1}{1-\Omega_K} - \frac{1}{1-\Omega_\mu} & & & \\ \frac{1}{1-\Omega_K} - \frac{1}{1-\Omega_\mu} & \frac{1}{1-\Omega_K} + \frac{2}{1-\Omega_\mu} & \frac{1}{1-\Omega_K} - \frac{1}{1-\Omega_\mu} & & 0 & \\ \frac{1}{1-\Omega_K} - \frac{1}{1-\Omega_\mu} & \frac{1}{1-\Omega_K} - \frac{1}{1-\Omega_\mu} & \frac{1}{1-\Omega_K} + \frac{2}{1-\Omega_\mu} & & & \\ \hline & & & \frac{3}{1-\Omega_\mu} & 0 & 0 \\ & 0 & & 0 & \frac{3}{1-\Omega_\mu} & 0 \\ & & & 0 & 0 & \frac{3}{1-\Omega_\mu} \end{array}\right] \tag{46}$$

which may be conditionally admissible if only $(1-\Omega_\mu)/(1-\Omega_K) = [(1-2v)/(1+v)]^{\frac{1}{2}}$.

6. Conclusions

In this study, the detailed dissections concentrated on isotropic double scalar damage variables are discussed. Based on the irreversible thermodynamics, the expressions of specific damage energy release rate, the coupled constitutive equations corresponding to damage and the conditions of admissibility for two scalar damage effective tensors have been developed in this paper. It is worth mentioning that the above-mentioned theoretical formulations are only logically reasonable. Owing to the limitations of time, conditions, funds, etc, they should be subject to multifaceted experiments before their innovative significance can be fully verified. As such, there is still room for further refinement of the theory of double scalar damage variables in future research work. The current level of research can only be regarded as an exploratory attempt in this field.

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