Vibration of axially functionally graded nano rods and beams with a variable nonlocal parameter

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Abstract. Vibration of axially functionally graded nano-rods and beams is investigated. It is assumed that the material properties change along the rod and beam length. The Ritz method with algebraic polynomials is used in the formulation of the problems. Stress gradient elasticity theory is utilized in order to include the nonlocal effects. Frequencies are obtained for different boundary conditions, geometrical and material properties. Nonlocal parameter is assumed as changing linearly or quadratically along the length of the nanostructure. Frequencies are compared to constant nonlocal parameter cases and considerable differences are observed between constant and variable nonlocal parameter cases. Mode shapes in various cases are depicted in order to explain the effects of axial grading.

Keywords: vibration; axially functionally graded; nanorod; nanobeam; nonlocal elasticity

1. Introduction

Functionally graded (FG) material is a compositional gradient of two or more component materials. Materials can be designed for specific applications with the variation of their composition and structure gradually. FG materials can have the desirable properties of each component because of a homogenous mixture of constituents.

Functionally graded material (FGM) structures have taken great interest of engineers recent years. They were designed for thermal isolation for aerospace structural applications and fusion reactors, especially. Metal-ceramic functionally graded materials are used in extremely high temperature environments as a structural element (Bharti *et al.* 2013).

Functionally graded nano rod and beam structures are used in some nano electromechanical system similar to their macro counterparts because of smooth variation of the material properties in preferred directions of structures. Although thickness wise grading is generally considered in the previous studies, axially functionally graded rods and beams can also be used at the nano length scale.

Continuous material property variation provides continuous stress distribution in the FG structures, where a discontinuous stress distribution appears in laminated composites. FGMs with

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material properties varying only in the thickness direction can be manufactured by the high speed centrifugal casting (Fukui 1990, Yamanouchi *et al*. 1990, Berger *et al*. 1994, Moya 1995) and those with properties varying in the plane of a sheet by the ultraviolent irradiation to alter chemical composition (Lambros *et al*. 1999). Studies related with structural analysis of FGM with properties varying in the thickness direction can be found in (Vel and Batra 2004, Qian *et al*. 2004, Ferreira *et al*. 2006, Uymaz and Aydogdu 2007, Aydogdu 2008, Şimşek 2010).

Some studies can also be obtained related to axially graded macro structures. Elishakoff and co-workers (Elishakoff and Guédé 2004, Wu *et al*. 2005, Elishakoff and Johnson 2005) studied vibration and buckling of inhomogeneous beams for some preselected polynomial mode shapes. Recently, (Anandakumar and Kim 2010) used the Rayleigh-Ritz method and the finite element method in analysis of free vibration of a three-dimensional functionally graded cantilever beam. Some specific material gradations are considered in the axial direction: Exponential, linear and hyperbolic tangent material gradations. (Huang and Li 2010) studied axially graded beams with non-uniform cross section in free vibration case using Fredholm integral equations. Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams investigated by (Shahba *et al*. 2011) on the classical and non-classical boundary conditions using a finite element formulation, recently. (Qian and Batra 2005) investigated optimal natural frequencies in bidirectional functionally graded plates. They obtained that functionally grading in axial direction provides good vibration results. Free (Janghorban and Zare 2011, Nejad and Hadi 2016), forced (Barati 2017) and nonlinear (Hosseini-Hashemi and Nazemnezhad 2013) vibrations of FGM nanostructures also studied by scientists. Dynamic instability (Sedighi *et al*. 2015) and magnetic-electric fields effect (Ebrahimi and Jafari 2017) investigated in detail.

Axially grading has been also considered in some of the previous nanoscale studies (Akgöz and Civalek 2013, Shafiei *et al*. 2016, Li *et al*. 2017, Shafiei *et al*. 2017). Ebrahimi and co-workers studied vibration of the axially graded nanobeams in thermal environment (Ebrahimi *et al*. 2017, Ebrahimi and Barati 2017a, c, Ebrahimi *et al*. 2018) and embedded in elastic medium (Ebrahimi and Barati 2017b, d). Using a bottom-to-up approach by suitable arranging atoms axially grading may also be possible artificially. Continuously graded thick Si and Ge nanowires along the its length shows local variations in Raman phonon bands (Yang *et al*. 2008).

Free longitudinal vibration of axially functionally graded tapered nanorods has been studied by (Şimşek 2012) using the Galerkin method. (Huang *et al*. 2013) have investigated the vibration of axially functionally graded Timoshenko beam with non-uniform cross section.

In the nanoscale continuum modeling of the nanostructures the stress gradient elasticity has been used. Nonlocal elasticity has been firstly proposed by (Eringen 1976, 1983) and has been used in statics and dynamics analysis of nanorods and beams (Peddieson *et al*. 2003, Sudak 2003, Wang 2005). Molecular dynamic simulations and nonlocal continuum models are compared for wave propagation in carbon nanotubes (Hu *et al*. 2008). Good agreement is observed between two results.

In the previous studies related to nano functionally graded materials, the nonlocal parameter is generally assumed as a constant. But similar to other material properties like elasticity modulus, density and Poisson ratio, nonlocal parameter should also be variable depending on the grading directions.

In the present study, the vibration of axially functionally graded nano rods and beams has been investigated by using the Classical Beam Theory (CBT). The Ritz method is used in the solution of vibration problems. A linear variation of material properties (Young modulus and density) is assumed in the length direction of beams. For nonlocal parameter linear and quadratic variations

are considered. It is assumed that the elasticity modulus, density and nonlocal parameter of rods and beams are changing in the axial direction. The frequency parameters are obtained for various material and geometrical properties, nonlocal parameter and boundary conditions.

2. Analysis

2.1 The nonlocal elasticity model for nano beam

An axially functionally graded beam of length L and radius R is considered. A cartesian coordinate system is chosen in which x axis in the length direction; z and y axes are cross sectional coordinates of the beam, respectively (Fig. 1). It is assumed that the material properties of the nano beam are changing along the length direction.

In the frame work of the Euler Bernoulli beam theory displacement field can be written as

$$U(x,y) = u(x,t) - zw_{,x}$$
$$V(x,y,t) = 0 \tag{1}$$
$$W(x,t) = w(x,t)$$

where U and W are the axial and transverse displacement of a typical point of the beam and u and w are the corresponding in-plane displacements, t is the time. It should be noted that deformation in the y direction is neglected. Assuming small deformations, linear strain components are

$$\varepsilon_x = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} = \varepsilon_x{}^0 + z\kappa$$
$$\varepsilon_x{}^0 = \frac{\partial u}{\partial x} \tag{2}$$
$$\kappa = \frac{\partial^2 w}{\partial x^2}$$

where ε is the axial strain and κ is the curvature. Stress and moment resultant of the beam, without any dimensional or material limitation, can be defined as

$$N = \int_A \sigma_{xx}\, dA, \qquad M = \int_A z\sigma_{xx}\, dA \tag{3}$$
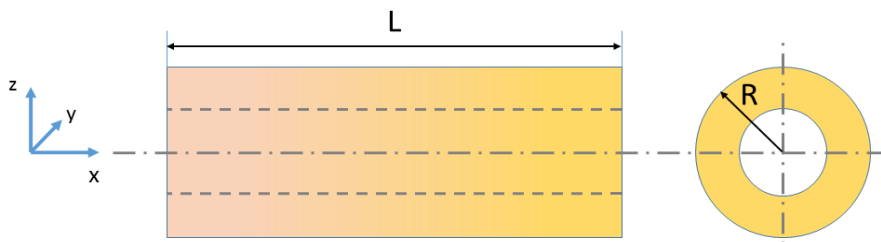

Fig. 1 FGM Nanostructure

where N and M are the force and moment resultants, respectively and A is the cross sectional area of the beam. Virtual displacement principle can be written as

$$\int_A \sigma_{ij}\, \delta\varepsilon_{ij}\, dV = \int_{S_2} \hat{t}_i\, \delta u_i\, dS_2 + \int_V f_i\, \delta u_i\, dV \tag{4}$$

where σ_{ij} is the components of the stress tensor, V is the volume, S_2 is the boundary, f_i is the body force, δ is the variational symbol and $\hat{t}_i$ is the surface stress. Using Eq. (1-3) in Eq. (4) following relation can be obtained

$$\int_0^{t_0}\int_0^L \left[N\delta\varepsilon_x{}^0 + M\delta\kappa - m_0\left(\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t}\right) - m_2\frac{\partial^2 w}{\partial x \partial t}\frac{\partial^2 \delta w}{\partial x \partial t} - f\delta u - q\delta w \right.$$
$$\left. - N_e\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} \right] dx\, dt = 0 \tag{5}$$

where $m_i = \int_A \rho z^i dA$ $(i = 0, 2)$ and f, q and N_e are the axial, flexural and in-plane external forces, respectively. Following equations of motion can be obtained from Eq. (5)

$$\frac{\partial N}{\partial x} + f = m_0(x)\frac{\partial^2 u}{\partial t^2} \tag{6a}$$

$$\frac{\partial^2 M}{\partial x^2} + q - \frac{\partial}{\partial x}\left(N\frac{\partial w}{\partial x}\right) = m_0(x)\frac{\partial^2 w}{\partial t^2} - m_2(x)\frac{\partial^4 w}{\partial x^2 \partial t^2} \tag{6b}$$

Here Eqs. (6a) and (6b) are the axial and transverse equations of motion of a beam structure. For uncoupled problems, Eq. (6a) is also called the classical rod equation of motion. Following boundary conditions are also obtained at $x = 0$ and $x = L$.

$$u \quad \text{or} \quad N \tag{7a}$$

$$w \quad \text{or} \quad \frac{\partial M}{\partial x} - N_e\frac{\partial w}{\partial x} + m_2\frac{\partial^3 w}{\partial x \partial t^2} = V_s \tag{7b}$$

$$-\frac{\partial w}{\partial x} \quad \text{or} \quad M \tag{7c}$$

where V_s is the shear force. The nonlocal constitutive relations of stress gradient elasticity has been proposed as (Lu *et al.* 2007, Aydogdu 2009a)

$$(1 - \mu\nabla^2)\tau_{kl} = \lambda\varepsilon_{rr}\delta_{kl} + 2G\varepsilon_{kl} \tag{8}$$

where τ_{kl} is the nonlocal stress tensor, ε_{kl} is the strain tensor, λ and G are the Lame constants, $\mu = (e_0 a)^2$ is called the nonlocal parameter, a is an internal characteristic length (nm) and e_0 is a constant. Physical interpretation and choice of this parameter has been discussed in some of the previous studies (Aydogdu 2009a, b).

In this study, $0 \leq (e_0 a)^2 \leq 2$ nm^2 is chosen in order to investigate nonlocality effects as suggested in (Wang 2005). It should be noted that for axially graded (or tapered) nano structure nonlocal parameter μ cannot be a constant. This is due to variation of the micro structure of the solid along the axial direction. It means that the atomic distance changes along the length of the nano structure or there are different atoms along the length of the structure. Therefore, the nonlocal parameter is assumed as a function of axial coordinate in this study.

2.2 Equation of motion of a nanorod

For the axial vibration of an axially functionally graded nanorod Eq. (8) can be written in the following one dimensional form

$$\left(1 - \mu(x)\frac{\partial^2}{\partial x^2}\right)\tau_{xx} = E(x)\varepsilon \tag{9}$$

where $E(x)$ is the modulus of elasticity. Integrating Eq. (9) with respect to the area of nanorod, and multiplying by z and integrating again with respect to area gives the following relations

$$N - \mu(x)\frac{\partial^2 N}{\partial x^2} = N^L \tag{10a}$$

$$M - \mu(x)\frac{\partial^2 M}{\partial x^2} = M^L \tag{10b}$$

where the local force and moment resultants are defined in the conventional manner

$$N^L = EA\varepsilon_{xx}{}^0 \tag{11a}$$

$$M^L = EI\kappa \tag{11b}$$

From Eqs. (4) and (6) and Eq. (8) axial force can be written as

$$N = EA\frac{\partial u}{\partial x} + \mu(x)\frac{\partial}{\partial x}\left(m_0(x)\frac{\partial^2 u}{\partial t^2} - f\right) \tag{12}$$

Using Eq. (8) in Eq. (6) gives the equation of motion for the axial free-vibrating nanorod in the nonlocal elasticity in terms of displacement

$$\frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left[\mu(x)\frac{\partial}{\partial x}\left(m_0(x)\frac{\partial^2 u}{\partial t^2} - f\right)\right] + f = m_0(x)\frac{\partial^2 u}{\partial t^2} \tag{13}$$

Eq. (13) is the consistent fundamental equation of the axially graded nonlocal rod model. This equation leads to second order variable coefficient ordinary differential equation. Unfortunately it is not possible to obtain a general solution of this kind of equations. When $\mu = 0$ nm^2, it is reduced to the equation of the classical rod model. When material properties are uniform and homogeneous along the length direction of nano rod previously obtained equations of motion of nonlocal

nanorod is obtained.

2.3 Equation of motion of a nanobeam

Using Eq. (6b) and (10b) the moment relation of nonlocal elasticity can be written as

$$M = -EI\frac{\partial^2 w}{\partial x^2} + \mu(x)\left[\frac{\partial}{\partial x}\left(N_e\frac{\partial w}{\partial x}\right) - q + m_0\frac{\partial^2 w}{\partial x^2} - m_2\frac{\partial^4 w}{\partial x^2 \partial t^2}\right] \tag{14}$$

Inserting Eq. (14) into Eq. (10b) leads to following equation of motion for a nanobeam

$$\frac{\partial^2}{\partial x^2}\left(-EI\frac{\partial^2 w}{\partial x^2} + \mu(x)\left[\frac{\partial}{\partial x}\left(N_e\frac{\partial w}{\partial x}\right) - q + m_0\frac{\partial^2 w}{\partial x^2} - m_2\frac{\partial^4 w}{\partial x^2 \partial t^2}\right]\right) + q - \frac{\partial}{\partial x}\left(N\frac{\partial w}{\partial x}\right)$$
$$= m_0\frac{\partial^2 w}{\partial x^2} - m_2\frac{\partial^4 w}{\partial x^2 \partial t^2} \tag{15}$$

Again, this equation is a variable coefficient differential equation. The equation of motion of the conventional Euler Bernoulli beam theory is obtained from Eq. (15) by setting $\mu = 0$.

2.4 Functionally graded materials

The functionally graded materials are produced by mixing two or more materials. In the present study variation of the material properties (elasticity modulus, density and nonlocal parameter) are assumed in the following forms (It should be noted that these material properties can be chosen as any function in the present formulation)

$$E(x) = \alpha_1 x + \alpha_2 \tag{16a}$$

$$\rho(x) = \beta_1 x + \beta_2 \tag{16b}$$

$$\mu(x) = \gamma_0 x^2 + \gamma_1 x + \gamma_2 \tag{16c}$$

where related parameters are

$$\alpha_1 = E_1 - E_0 \quad , \quad \alpha_2 = E_0 \tag{17a}$$

$$\beta_1 = \rho_1 - \rho_0, \quad \beta_2 = \rho_0 \tag{17b}$$

$$\gamma_0 = \gamma_1 = \mu_1/2, \quad \alpha_2 = \mu_0 \tag{17c}$$

here E_0, ρ_0, μ_0 and E_1, ρ_1, μ_0 are the material properties at the left and right end of the beam respectively. Material property variations are shown in Fig. 2.

2.5 Ritz formulation

Analytical solution of equations of motion of axially functionally graded nanorods and beams are difficult or not possible, therefore an approximate variational method, Ritz method will be used
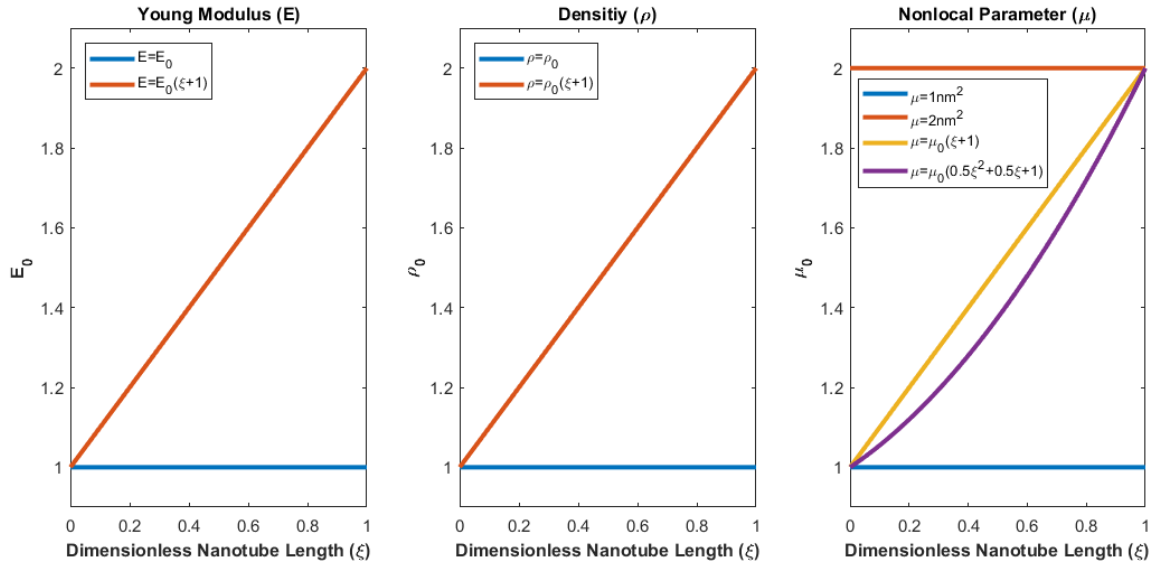
Fig. 2 Material properties of FGM nanostructures

in the formulation of the present problem.

2.5.1 Nano rod

The strain energy of an axially functionally graded rod can be (Adali 2008, 2009, 2015, Robinson and Adali 2016)

$$U_s = \frac{1}{2} \int_0^L E(x)\, A u_{,x}^2 \, dx \tag{18}$$

where *u* and *A* are the displacement in the *x* direction and cross-sectional area respectively and L is the length of the rod. The kinetic energy of the rod is (Adali 2008, 2009, 2015, Robinson and Adali 2016)

$$T = \frac{1}{2} \int_0^L \left(m(x)\ddot{w}^2 + \mu(x)m(x)\ddot{w}_{,x}^2 \right) dx \tag{19}$$

where a dot denotes derivative with respect to time. In this relation the second term of the kinetic energy is due to the nonlocal elasticity.

2.5.2 Nano beam

A functionally graded beam is an inhomogeneous structure consisting of a mixture of isotropic materials. Therefore, for thin beams say $L/h > 20$ classical beam theory can be used. Since axially grading leads to a variable coefficient differential equation for vibration of beams, the Ritz method (Aydogdu and Filiz 2011) is used in the present study. Strain energy and kinetic energy of the beam for the Ritz formulation can be defined as below. The strain energy of the beam

$$U_s = \frac{1}{2} \int_0^L IE(x)\, w_{,xx}{}^2\, dx \tag{20}$$

where w and I are the displacement in the z direction and moment of inertia of cross-section respectively and L is the length of the beam. The kinetic energy of the beam

$$T = \frac{1}{2} \int_0^L \left(m(x)\ddot{w}^2 + \mu(x)m(x)\ddot{w}_{,x}{}^2 \right) dx \tag{21}$$

where dot denotes the time derivative. In the Ritz method, displacement component is defined in the following form

$$w(x,t) = \sin \omega t \sum_{i=i_0}^{I} A_i \psi_i(x) \tag{22}$$

where A_i's are the unknown coefficients, ω is the angular frequency and $\psi_i(x)$ is a function which satisfies at least geometric boundary conditions of the beam. Convergence of this function is guaranteed if this function is mathematically complete set. To determine the vibration frequencies of the axially graded beams following functional is defined

$$F = T_{\max} - U_{\max} \tag{23}$$

This functional should be minimized with respect to unknown coefficients given in Eq. (22)

$$\frac{\partial F}{\partial A_i} = 0 \tag{24}$$

This yields a total of $I \times I$ simultaneous, linear, homogeneous equations in an equal number of unknowns A_i. Those equations can be described as an eigen-value problem

$$([K] - \Omega^2[M])\Delta = 0 \tag{25}$$

where K and M are the stiffness and mass matrix respectively; Ω is the dimensionless frequency parameter defined as $\Omega^4 = (m_0 \omega^2 L^4)/(E_0 I)$ and $\Omega^2 = (m_0 \omega^2 L^2)/(E_0 A)$ for the nano beam and the nano rod respectively and Δ is the column vector of unknown coefficients A_i. Here ω (rad/s) is the angular frequency of vibrations. The mode shapes corresponding to any Ω is found by substituting that value into Eq. (25) and solving for the eigenvector components A_i/A_1. Inserting these components into Eq. (22) gives mode shape of nanorod or nano beam. In the present study, mode shape amplitudes are normalized with respect to highest amplitude value of the corresponding mode, therefore the highest amplitude in each mode equals to unity.

After the defining non-dimensional coordinate ($\xi = x/L$), the following simple algebraic polynomials are used

$$w(\xi,t) = \sum_{i=i_0}^{M} A_i X_i(\xi) \sin \omega t \tag{26}$$

where the polynomial is defined as

$$X_i = \xi^i (\xi - 1)^B \tag{27}$$

and $A_i's$ are unknown undetermined coefficients. The values of $B = 0$, 1 and 2 correspond to the free, hinged and clamped edge, respectively. It should be noted that rod kinetic and strain energy equations (Eqs. (18)-(19)) should be used in Eq.(23) when solving rod vibration problem.

3. Results and discussions

Investigation on vibration of axially functionally graded nano rods and nano beams with various boundary conditions are carried out. Following boundary conditions are assumed in the present study.

Nanorod: Clamped-Clamped (C-C), Clamped-Free (C-F)
Nanobeam: Clamped-Free (C-F), Clamped-Simply Supported (C-SS)

In the Ritz method, displacement field should be chosen as satisfy at least the geometric boundary conditions. The kinematic boundary conditions of the nanorod and nanobeam are given in Table 1. It should be noted that free edge boundary conditions are approximately satisfied in this study.

In order to see the effect of nonlocal parameter five different nonlocal parameter values are chosen. For axially functionally graded rods and beams $\mu = 1$ nm^2 and $\mu = 2$ nm^2 are assumed at the left and right end of nano structure, respectively. The variations of μ parameter are assumed in the following forms.

$$
\begin{aligned}
&\mu = 0 \text{ (local)} \\
&\mu = 1 \text{ nm}^2 \text{ (nonlocal with constant } \mu) \\
&\mu = 2 \text{ nm}^2 \text{ (nonlocal with constant } \mu) \\
&\mu = \mu_0 (\xi + 1) \text{ (nonlocal with linear } \mu \text{ variation)} \\
&\mu = \mu_0 (0.5\xi^2 + 0.5\xi + 1) \text{ (nonlocal with quadratic } \mu \text{ variation)}
\end{aligned}
\tag{28}
$$

Sample convergence studies are carried out for the non-dimensional frequency parameter Ω and

Table 1 Kinematic boundary conditions of rods and beams

	Nanorod			Nanobeam	
	$\xi = 0$	$\xi = 1$		$\xi = 0$	$\xi = 1$
C-C	$u = 0$	$u = 0$	C-F	$w = 0$ $\dfrac{\partial w}{\partial \xi} = 0$	$\dfrac{\partial^2 w}{\partial \xi^2} = 0$ $\dfrac{\partial^3 w}{\partial \xi^3} = 0$
C-F	$u = 0$	$\dfrac{\partial u}{\partial \xi} = 0$	C-SS	$w = 0$ $\dfrac{\partial w}{\partial \xi} = 0$	$w = 0$ $\dfrac{\partial^2 w}{\partial \xi^2} = 0$

Table 2 Convergence of non-dimensional frequencies for nano rods and nano beams
($L = 5$ nm, $\mu = (0.5\xi^2 + 0.5\xi + 1)$)

N	Nanorod C-C			Nanorod C-F			Nanobeam C-F			Nanobeam C-S		
	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
3	2.479	3.385	3.965	1.304	3.137	3.577	1.589	3.419	6.068	3.324	5.170	7.790
4	2.479	3.385	3.668	1.304	3.093	3.531	1.589	3.419	5.198	3.323	5.133	6.609
5	2.479	3.384	3.601	1.304	3.093	3.531	1.589	3.418	5.162	3.323	5.118	6.447
6	2.479	3.384	3.577	1.304	3.093	3.531	1.589	3.418	5.145	3.323	5.116	6.406
7	2.479	3.384	3.565	1.304	3.093	3.531	1.589	3.418	5.144	3.323	5.116	6.394

Table 3 Convergence of non-dimensional frequencies for nano rods and nano beams
($L = 30$ nm, $\mu = (0.5\xi^2 + 0.5\xi + 1)$)

N	Nanorod C-C			Nanorod C-F			Nanobeam C-F			Nanobeam C-S		
	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
3	3.098	6.239	9.339	1.359	4.628	9.753	1.682	4.516	10.296	3.804	6.919	13.103
4	3.098	6.087	9.320	1.359	4.583	7.759	1.682	4.480	7.613	3.803	6.918	9.963
5	3.098	6.086	8.831	1.359	4.569	7.545	1.682	4.475	7.584	3.803	6.892	9.945
6	3.098	6.085	8.829	1.359	4.569	7.469	1.682	4.475	7.521	3.803	6.982	9.817
7	3.098	6.085	8.817	1.359	4.569	7.464	1.682	4.475	7.521	3.803	6.892	9.816

tabulated in Tables 2 and 3. It is seen that the convergence is better for the lower modes of vibration and long nano rods and beams. The highest difference between 6 and 7 terms is 0.33% for the third mode of C-C nanorod at $L = 5$ nm. The non-dimensional frequency parameters obtained with 6 and 7 term are very close to each other. Consequently, all of the remaining results presented have been obtained using 7 terms in Eq. (26).

In Table 4, the present non-dimensional frequency parameters are compared with classical analytical results for axially graded rods and beams for different boundary conditions. The maximum percentage difference is less than 3%. It can be concluded that, there is a good agreement between the results.

After verifying the convergence and accuracy of the present Ritz analysis, the non-dimensional frequency parameters are obtained for axially graded nano rods and nano beams with different material properties, nonlocal parameter and boundary conditions. In Tables 5-8 percentage errors

Table 4 Comparison of Non-Dimensional Frequencies for linear material property variations
($E = (-\xi + 2)$, $\rho = (-\xi + 2)$, $\mu = 0$)

	Nanorod C-C		Nanorod C-F		Nanobeam C-F	
	Present	(Aydogdu and Filiz 2011)	Present	(Aydogdu and Filiz 2011)	Present	(Aydogdu and Filiz 2011)
Ω_1	3.123	3.1235	1.794	1.7932	2.077	2.0245
Ω_2	6.273	6.2748	4.802	4.8038	4.849	4.715
Ω_3	9.418	9.4580	7.909	7.9629	7.950	7.738

of non-dimensional frequencies using constant nonlocal parameters are compared with that of variable nonlocal linear and parabolic parameters. Percentage errors are computed according to the following equation

$$Error\ \% = \frac{\Omega_{constant\ ,\mu} - \Omega_{variable\ ,\mu}}{\Omega_{variable\ ,\mu}} x100 \tag{29}$$

From these tables, it is seen that non-dimensional frequencies obtained by using $\mu = 1$ nm^2 are always lower than non-dimensional frequencies obtained by using variable μ, whereas the reverse is true for $\mu = 2$ nm^2. $\mu = 1.5$ nm^2 results are lower than variable case results except some frequencies of C-F nano rods and C-S nano beams. The percentage errors are higher in the rod case when compared with the beam vibration case. Also the errors are increases with increasing the mode number. Highest errors are obtained with more constrained boundary conditions (CC rod). Errors are changing between 0.686% and 24.947% for the rod problem and between 0.339% and 9.225% for the beam problem when $\mu = 1$ nm^2.

When $\mu = 2$ nm^2, the errors are between -2.348% and -9.055% for the rod problem and between -0.994% and -6.608% for the beam problem. It is natural to assume the nonlocal parameter μ as an average of the end values. So in the present problem average value of the μ is 1.5 nm^2. The percentage errors are higher than 5% for some cases and these results suggest to use a variable nonlocal parameter for the axially graded nano rods and nano beams.

In Figs. (3)-(6) the variation of dimensionless frequency parameter with μ_0 is given. It is seen that the dimensionless frequency parameters are decreasing with the increasing nonlocal parameter. Higher order frequencies are more sensitive to the nonlocal elasticity. Order of frequencies is obtained as local, non-local ($\mu = 1$ nm^2), non-local (quadratic), non-local (linear) and non-local ($\mu = 2$ nm^2). The higher order frequencies of the variable nonlocal parameters are much closer to

Table 5 Percentage frequency errors of constant nonlocal parameters compared to variable nonlocal parameters for C-C Nanorod ($L = 5$ nm)

	$\mu = 1$ nm^2		$\mu = 1.5$ nm^2		$\mu = 2$ nm^2	
	Linear error (%)	Quadratic error (%)	Linear error (%)	Quadratic error (%)	Linear error (%)	Quadratic error (%)
Ω_1	7.576	6.839	0.740	0.050	-4.938	-5.589
Ω_2	17.123	15.513	2.497	1.087	-7.737	-9.006
Ω_3	24.947	23.874	5.976	5.065	-6.351	-7.156

Table 6 Percentage frequency errors of constant nonlocal parameters compared to variable nonlocal parameters for C-F nanorod ($L = 5$ nm)

	$\mu = 1$ nm^2		$\mu = 1.5$ nm^2		$\mu = 2$ nm^2	
	Linear error (%)	Quadratic error (%)	Linear error (%)	Quadratic error (%)	Linear error (%)	Quadratic error (%)
Ω_1	0.962	0.686	-0.734	-1.005	-2.348	-2.615
Ω_2	11.757	10.014	0.699	-0.870	-7.614	-9.055
Ω_3	20.694	19.277	3.693	2.476	-7.691	-8.774

Table 7 Percentage frequency errors of constant nonlocal parameters compared to variable nonlocal parameters for C-F nanobeam ($L = 5$ nm)

	$\mu = 1$ nm^2		$\mu = 1.5$ nm^2		$\mu = 2$ nm^2	
	Linear error (%)	Quadratic error (%)	Linear error (%)	Quadratic error (%)	Linear error (%)	Quadratic error (%)
Ω_1	2.360	2.077	0.618	0.339	-0.994	-1.268
Ω_2	8.688	8.091	2.465	1.902	-2.137	-2.675
Ω_3	8.998	7.961	1.295	0.331	-4.181	-5.094

Table 8 Percentage frequency errors of constant nonlocal parameters compared to variable nonlocal parameters for C-S nanobeam ($L = 5$ nm)

	$\mu = 1$ nm^2		$\mu = 1.5$ nm^2		$\mu = 2$ nm^2	
	Linear error (%)	Quadratic error (%)	Linear error (%)	Quadratic error (%)	Linear error (%)	Quadratic error (%)
Ω_1	5.018	4.570	1.208	0.777	-2.004	-2.421
Ω_2	7.956	6.864	0.861	-0.158	-4.363	-5.330
Ω_3	9.225	7.882	0.577	-0.659	-5.445	-6.608

$\mu = 2$ nm^2 curve. The variable nonlocal parameter is more effective for the C-F boundary conditions when compared to the C-C boundary conditions. The quadratic nonlocal parameter variation gives higher results then the linear variation especially for the lower modes of vibration.
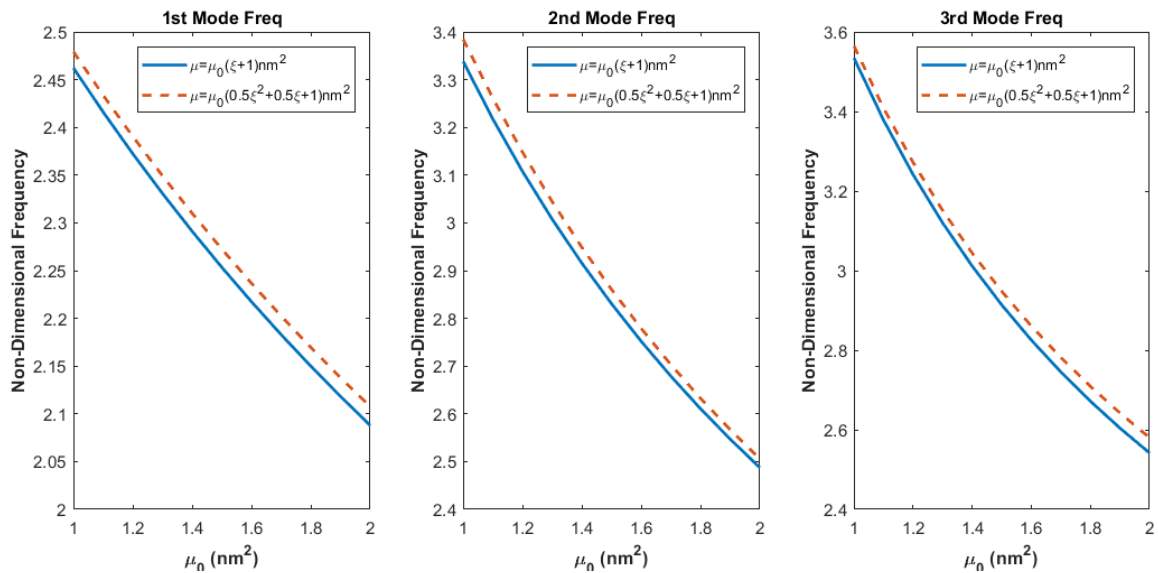
Fig. 3 Variation of first three non-dimensional frequency parameter with nonlocal parameter for variable nonlocal parameters for C-C nanorod ($L = 5$ nm)
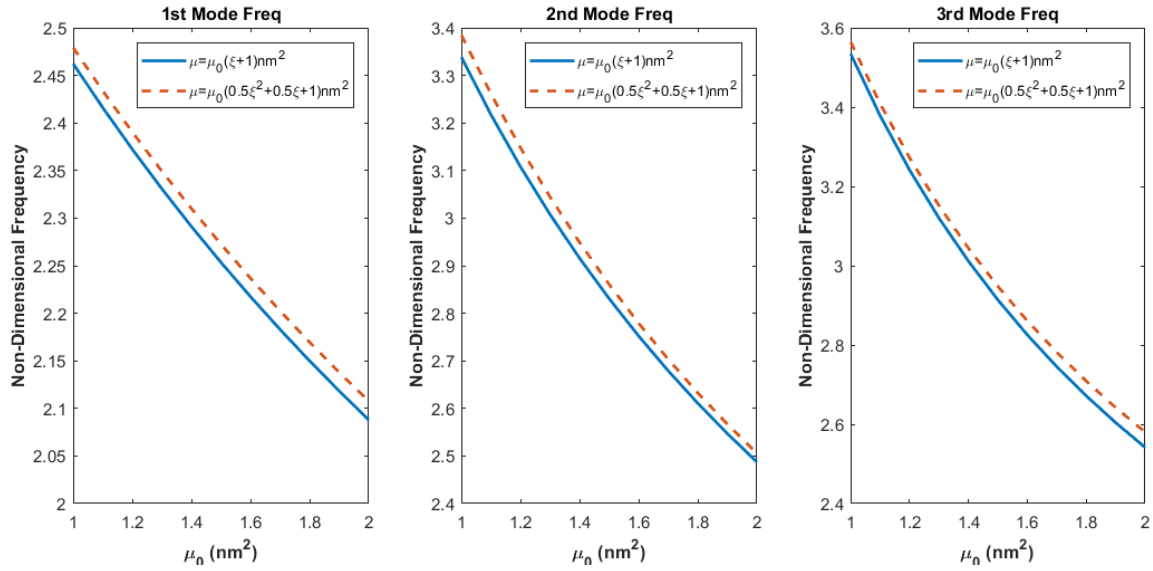
Fig. 4 Variation of first three non-dimensional frequency parameter with nonlocal parameter
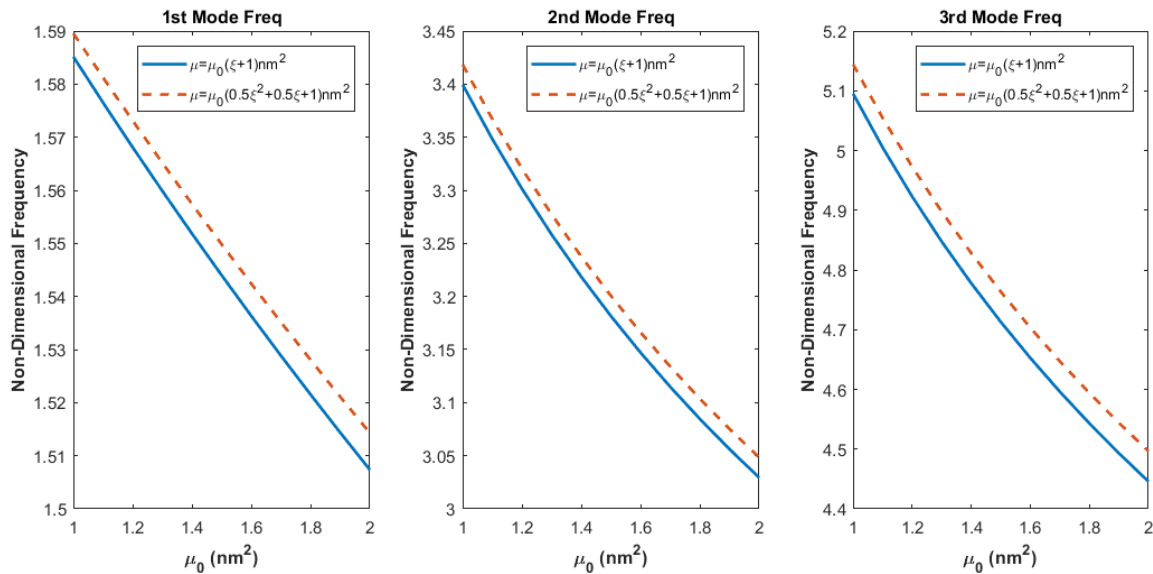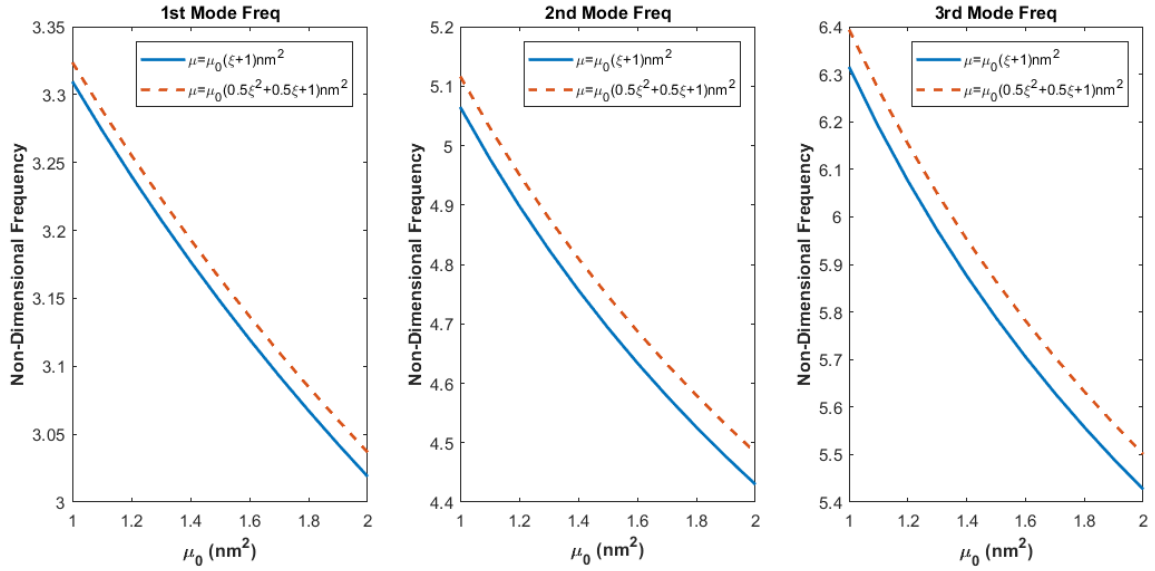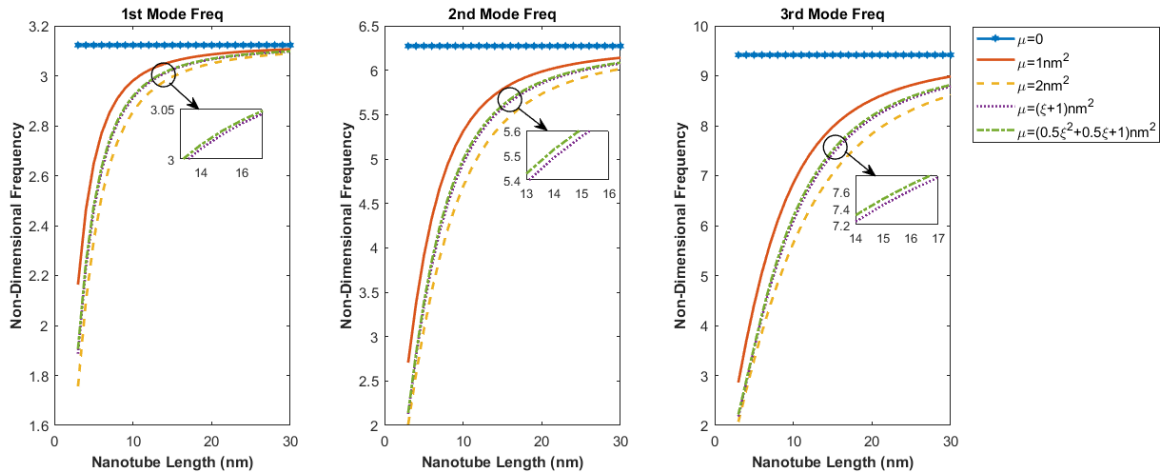for variable nonlocal parameters for C-F nanorod ($L = 5$ nm)

Fig. 5 Variation of first three non-dimensional frequency parameter with nonlocal parameter for
variable nonlocal parameters for C-F nanobeam ($L = 5$ nm)

The variation of the non-dimensional frequency parameters with length of nano structure are shown in Figs. 7-10. It is seen that with decreasing nanotube length difference between the classical and the nonlocal model increases. The difference between different μ_0 models is more pronounced for the lower nanotube lengths. The variable μ predicts frequencies between two

Fig. 6 Variation of first three non-dimensional frequency parameter with nonlocal parameter for variable nonlocal parameters for C-S nanobeam ($L = 5$ nm)

Fig. 7 Variation of first three non-dimensional frequency parameter with nano tube length L for various nonlocal parameters for C-C nanorod

constant values predicted. Frequencies are closer to $\mu_0 = 2$ nm^2 value especially for the beam problems. Also the difference between the classical elasticity and the nonlocal elasticity is more pronounced for the higher modes of vibration.

Normalized mode shapes equations (highest amplitude equals to unity) of nano-rods and beams obtained from Eq. (25) are given below for various boundary conditions and isotropic and FGM nano rods and nano beams.

C-C nanorod

$$w_{ISO,1} = -3.141\xi - 0.0002\xi^2 + 5.170\xi^3 - 0.022\xi^4 - 2.462\xi^5 - 0.200\xi^6$$
$$+ 0.875\xi^7 - 0.218\xi^8$$
$$w_{ISO,2} = -6.314\xi + 0.756\xi^2 + 32.723\xi^3 + 46.041\xi^4 - 210.184\xi^5$$
$$+ 191.767\xi^6 - 54.790\xi^7 + 0\xi^8$$
$$w_{ISO,3} = -9.586\xi + 7.101\xi^2 + 37.765\xi^3 + 676.543\xi^4 - 3011.278\xi^5$$
$$+ 4631.704\xi^6 - 3109.667\xi^7 + 777.416\xi^8$$
$$w_{FGM,1} = -3.462\xi + 1.449\xi^2 + 3.514\xi^3 - 0.722\xi^4 - 0.812\xi^5 + 0.011\xi^6 \tag{30}$$
$$- 0.006\xi^7 + 0.028\xi^8$$
$$w_{FGM,2} = -4.884\xi + 1.893\xi^2 + 9.963\xi^3 + 25.986\xi^4 - 78.819\xi^5$$
$$+ 96.576\xi^6 - 73.4710\xi^7 + 22.755\xi^8$$
$$w_{FGM,3} = -6.926\xi + 81.283\xi^2 - 845.196\xi^3 + 4254.318\xi^4 - 10976.366\xi^5$$
$$- 15275.630\xi^6 - 10904.819\xi^7 + 3122.076\xi^8$$

where subscript "ISO" denotes the isotropic case whereas "FGM" denotes the functionally graded case and "*i*" denotes the mode number:

C-F nanorod

$$w_{ISO,1} = 1.570\xi - 0.000038\xi^2 - 0.645\xi^3 - 0.0013\xi^4 + 0.082\xi^5 - 0.0030\xi^6$$
$$- 0.0032\xi^7$$
$$w_{ISO,2} = 4.721\xi - 0.270\xi^2 - 14.917\xi^3 - 10.882\xi^4 + 43.998\xi^5 - 29.649\xi^6 + 6\xi^7$$
$$w_{ISO,3} = 7.623\xi + 5.537\xi^2 - 119.540\xi^3 + 92.236\xi^4 + 270.585\xi^5 - 406.551\xi^6$$
$$+ 151.108\xi^7 \tag{31}$$
$$w_{FGM,1} = 1.973\xi - 0.950\xi^2 + 0.052\xi^3 - 0.185\xi^4 + 0.158\xi^5 - 0.059\xi^6 + 0.011\xi^7$$
$$w_{FGM,2} = 4.451\xi - 1.453\xi^2 - 10.579\xi^3 + 3.834\xi^4 - 0.691\xi^5 + 5.747\xi^6 - 2.310\xi^7$$
$$w_{FGM,3} = 2.323\xi + 1.858\xi^2 - 29.331\xi^3 + 74.077\xi^4 - 128.509\xi^5 + 121.740\xi^6$$
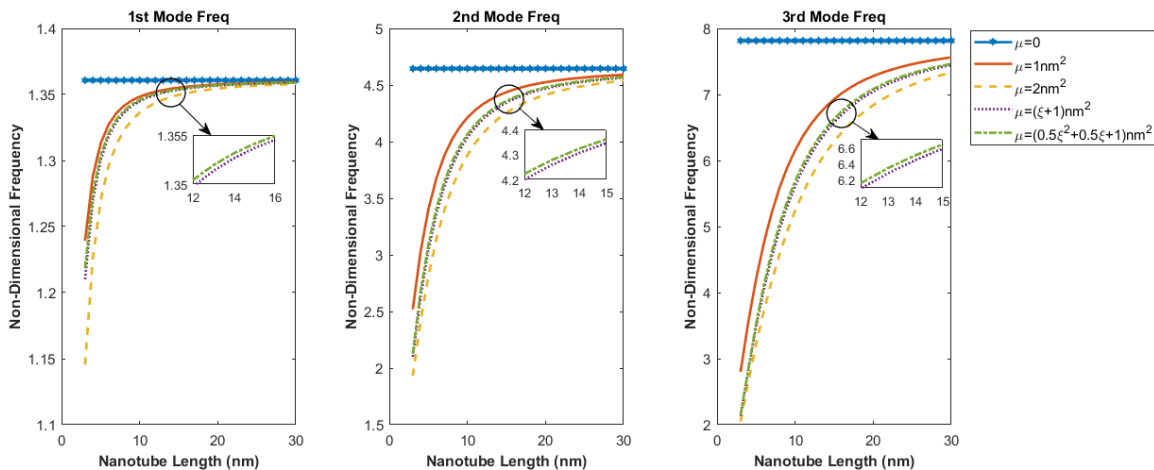$$- 41.159\xi^7$$

Fig. 8 Variation of first three non-dimensional frequency parameter with nano tube length *L* for various nonlocal parameters for C-F nanorod

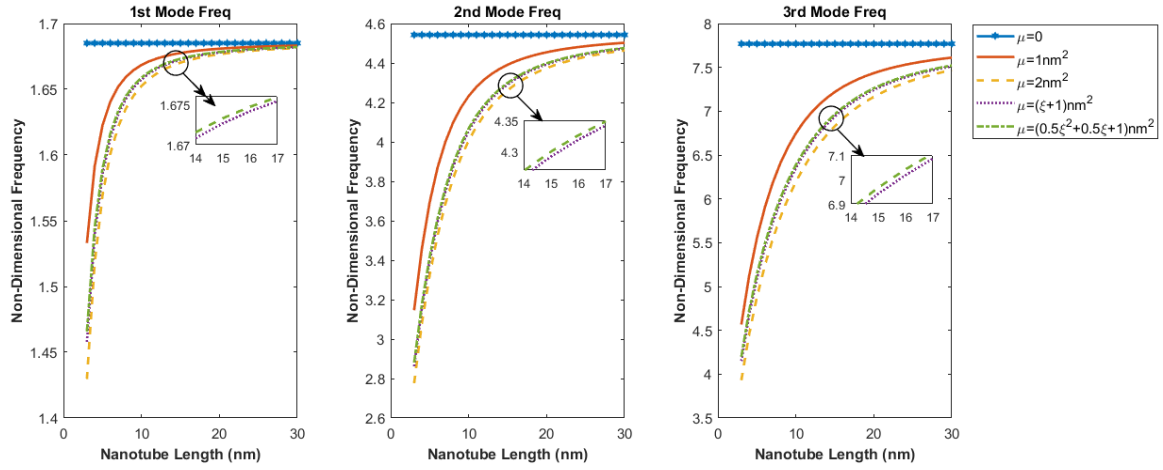*Metin Aydogdu, Mustafa Arda and Seckin Filiz*

Fig. 9 Variation of first three non-dimensional frequency parameter with nano tube length *L* for various nonlocal parameters for C-F nanobeam
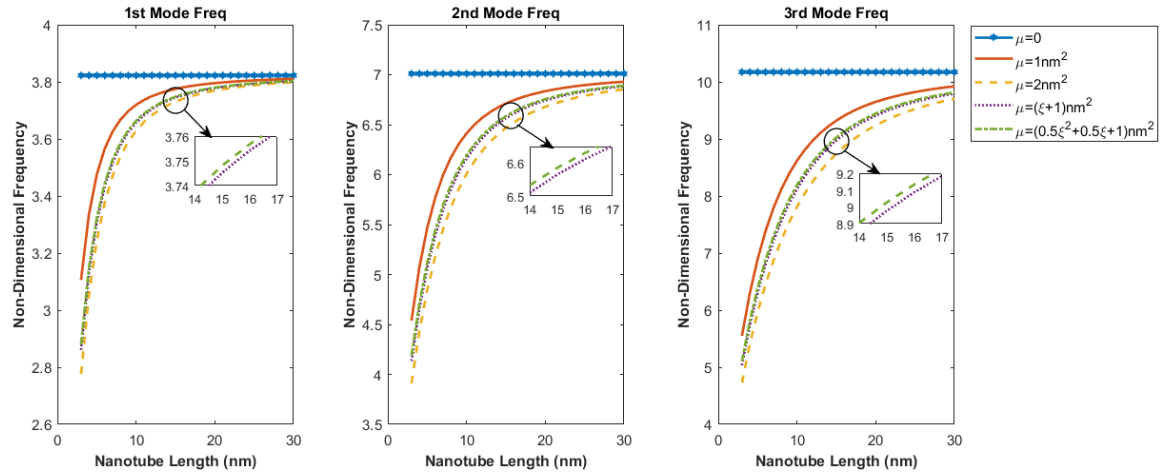

Fig. 10 Variation of first three non-dimensional frequency parameter with nano tube length *L* for various nonlocal parameters for C-S nanobeam

C-F nanobeam

$$
\begin{aligned}
w_{ISO,1} &= 1.614\xi^2 - 0.573\xi^3 - 0.095\xi^4 + 0.018\xi^5 + 0.045\xi^6 - 0.009\xi^7 + 0.0002\xi^8 \\
w_{ISO,2} &= 10.589\xi^2 - 10.630\xi^3 - 8.910\xi^4 + 2.530\xi^5 + 14.493\xi^6 - 11.110\xi^7 \\
&\quad + 2.334\xi^8 \\
w_{ISO,3} &= 22.556\xi^2 - 11.618\xi^3 - 162.909\xi^4 + 240.779\xi^5 - 0.189\xi^6 - 150.696\xi^7 \\
&\quad + 62.514\xi^8 \\
w_{FGM,1} &= 1.946\xi^2 - 1.372\xi^3 + 0.633\xi^4 - 0.350\xi^5 + 0.199\xi^6 - 0.068\xi^7 + 0.010\xi^8 \\
w_{FGM,2} &= 12.740\xi^2 - 20.186\xi^3 + 4.535\xi^4 - 1.279\xi^5 + 8.305\xi^6 - 5.840\xi^7 \\
&\quad + 1.123\xi^8
\end{aligned}
\tag{32}
$$

$$w_{FGM,3} = 25.664\xi^2 - 42.702\xi^3 - 55.419\xi^4 + 81.520\xi^5 + 100.678\xi^6 - 167.920\xi^7 \\ + 58.441\xi^8 \tag{32}$$

C-S nanobeam

$$w_{ISO,1} = -8.868\xi^2 + 8.035\xi^3 + 7.109\xi^4 - 3.225\xi^5 - 7.093\xi^6 + 4.235\xi^7 \\ + 0.156\xi^8 - 0.349\xi^9 \tag{33}$$
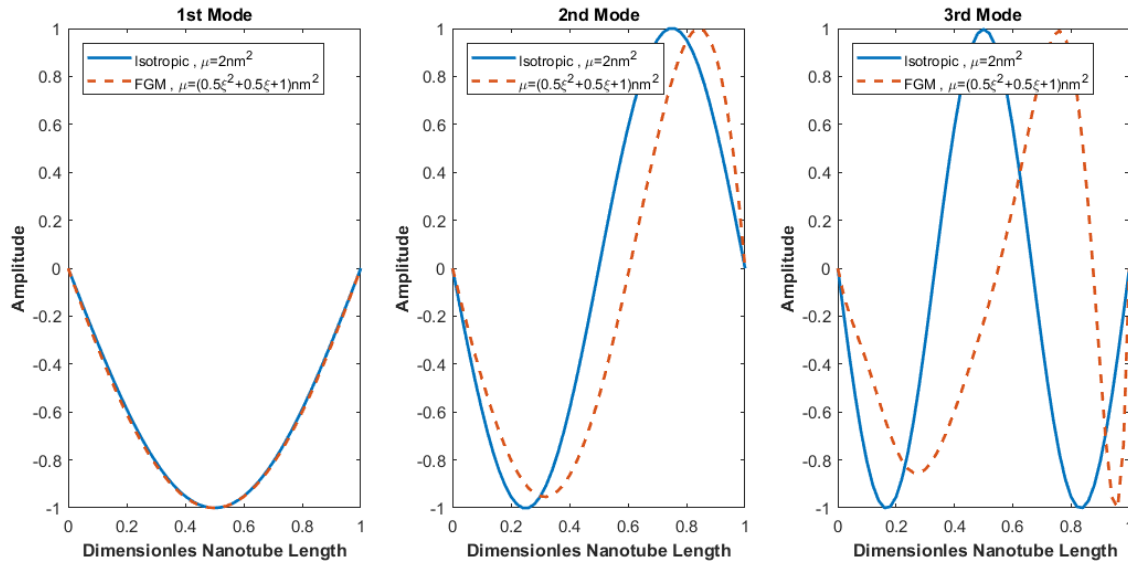

Fig. 11 First three mode shapes of C-C isotropic and axially functionally graded nanorods
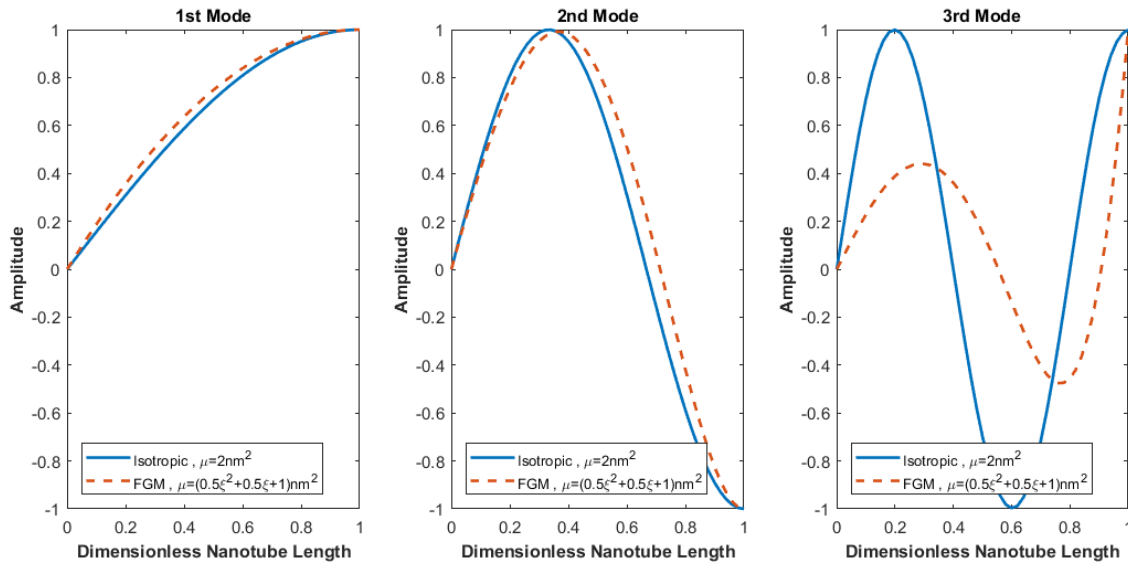

Fig. 12 First three mode shapes of C-F isotropic and axially functionally graded nanorods
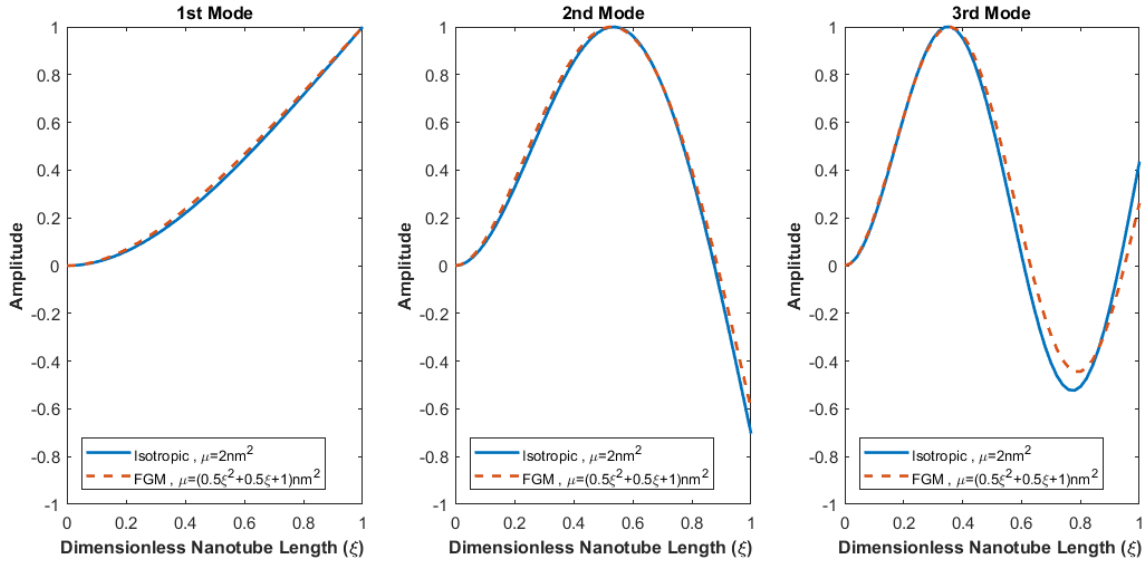
Fig. 13 First three mode shapes of C-F isotropic and axially functionally graded nanobeams
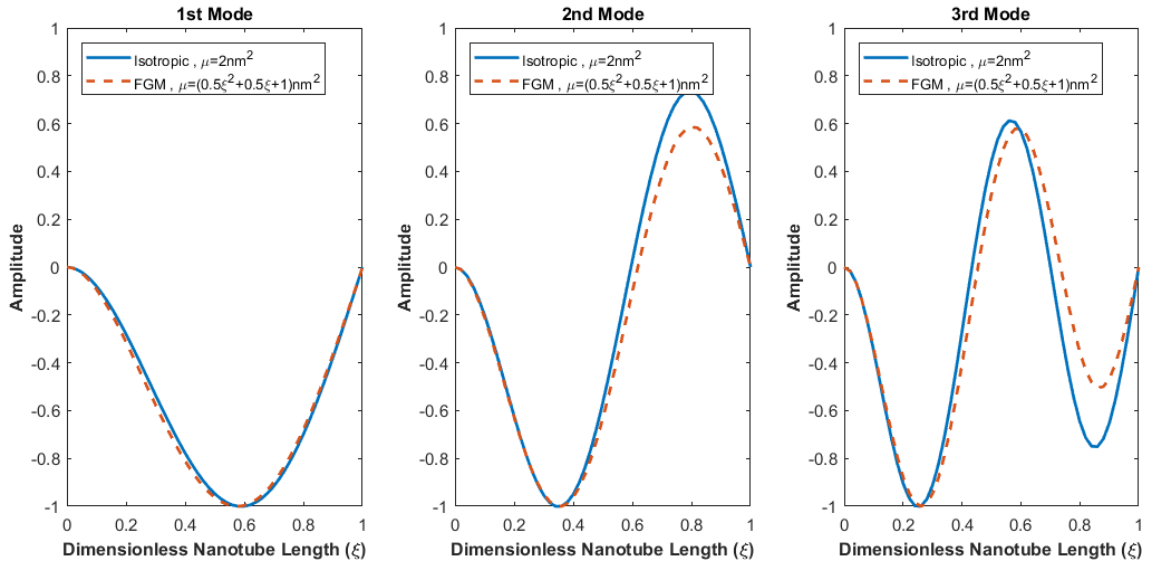

Fig. 14 First three mode shapes of C-S isotropic and axially functionally graded nanobeams

$$
\begin{aligned}
w_{ISO,2} &= -23.639\xi^2 + 29.713\xi^3 + 45.470\xi^4 + 141.851\xi^5 - 684.748\xi^6 \\
&\quad + 843.902\xi^7 - 435.520\xi^8 + 82.969\xi^9 \\
w_{ISO,3} &= -40.325\xi^2 - 13.487\xi^3 + 814.222\xi^4 - 1550.392\xi^5 - 423.082\xi^6 \\
&\quad + 3420.276\xi^7 - 3085.544\xi^8 + 878.332\xi^9 \\
w_{FGM,1} &= -10.969\xi^2 + 16.359\xi^3 - 3.664\xi^4 + 0.637\xi^5 - 5.002\xi^6 + 2.897\xi^7 \\
&\quad - 0.124\xi^8 - 0.134\xi^9
\end{aligned}
\tag{33}
$$

$$w_{FGM,2} = -26.338\xi^2 + 51.458\xi^3 + 3.508\xi^4 + 83.137\xi^5 - 391.421\xi^6$$
$$+ 459.133\xi^7 - 213.771\xi^8 + 34.292\xi^9$$
$$w_{FGM,3} = -44.591\xi^2 + 87.371\xi^3 + 152.017\xi^4 + 305.968\xi^5 - 3008.848\xi^6$$
$$+ 5251.263\xi^7 - 3677.971\xi^8 + 934.790\xi^9$$

(33)

Some typical mode shapes are shown in Figs. 11-14 for different boundary conditions and isotropic and quadratically graded nanorods and nanobeams. It is seen that the first mode shapes are approximately identical. The difference between the mode shapes of isotropic and axially FG rods and beams are more apparent for the higher modes and at the right end of the nanorod or nano beam. All the materials properties, young modulus, density and nonlocal parameter increase to the right end of the structure and this makes the structure more flexible at the right end. Considerable changes are observed for the nodes of the mode shapes. This gives an opportunity to the designer to tailor the mode shapes of the nano structures.

4. Conclusions

In the present study, the vibration of axially functionally graded nanorods and nanobeams was presented. The material properties are graded in the axial direction of the rod and beam linearly. The Euler-Bernoulli beam theory and Ritz method are employed for the analysis. The dimensionless frequency parameters are obtained for various material properties. The following main conclusions can be drawn from the present analysis.

- The percentage errors of using a constant nonlocal parameter leads to errors higher than 5% for some cases and these results suggest using a variable nonlocal parameter for the axially graded nano rods and nano beams. Errors are more apperent for the higher modes of vibration.
- Mode shapes of the axially functionally graded nanorods and nano beams are different from the isotropic nanorods and nanobeams and this may give various design opportunities.
- The Ritz method can be easily used in the formulation of axially functionally graded nanorods and beams with the variable nonlocal parameter.

One could easily vary the frequency parameters of the FG beam by varying the volume fraction of the constituents. Present study can be extended to the statics and dynamics analysis of nano plates or nano shells.

References

Adali, S. (2008), "Variational principles for multi-walled carbon nanotubes undergoing buckling based on nonlocal elasticity theory", *Phys. Lett.*, *Section A*: *General Atom. Solid State Phys.*, **372**(35), 5701-5705. DOI: 10.1016/j.physleta.2008.07.003

Adali, S. (2009), "Variational principles for transversely vibrating multiwalled carbon nanotubes based on nonlocal euler-bernoulli beam model", *Nano Lett.*, **9**(5), 1737-1741. DOI: 10.1021/nl8027087

Adali, S. (2015), "Variational Principles for Vibrating Carbon Nanotubes Conveying Fluid, Based on the Nonlocal Beam Model", *East Asian J. Appl. Math.*, **5**(3), 209-221. DOI: 10.4208/eajam.130814.250515a

Akgöz, B. and Civalek, Ö. (2013), "Free vibration analysis of axially functionally graded tapered Bernoulli-

Euler microbeams based on the modified couple stress theory", *Compos. Struct.*, **98**, 314-322.
DOI: 10.1016/j.compstruct.2012.11.020

Anandakumar, G. and Kim, J.H. (2010), "On the modal behavior of a three-dimensional functionally graded cantilever beam: Poisson's ratio and material sampling effects", *Compos. Struct.*, **92**(6), 1358-1371.
DOI: 10.1016/j.compstruct.2009.11.020

Aydogdu, M. (2008), "Semi-inverse method for vibration and buckling of axially functionally graded beams", *J. Reinf. Plast. Compos.*, **27**(7), 683-691. DOI: 10.1177/0731684407081369

Aydogdu, M. (2009a), "A general nonlocal beam theory: Its application to nanobeam bending, buckling and vibration", *Physica E: Low-Dimen. Syst. Nanostruct.*, **41**(9), 1651-1655.
DOI: 10.1016/j.physe.2009.05.014

Aydogdu, M. (2009b), "Axial vibration of the nanorods with the nonlocal continuum rod model", *Phys. E: Low-Dimen. Syst. Nanostruct.*, **41**(5), 861-864. DOI: 10.1016/j.physe.2009.01.007

Aydogdu, M. and Filiz, S. (2011), "Vibration analysis of piecewise and continuously axially graded rods and beams", In: Galloway AL (ed.) Mechanical Vibrations: Types, Testing and Analysis, Nova Publishers, pp. 95-146

Barati, M.R. (2017), "Nonlocal-strain gradient forced vibration analysis of metal foam nanoplates with uniform and graded porosities", *Adv. Nano Res., Int. J.*, **5**(4), 393-414. DOI: 10.12989/anr.2017.5.4.393

Berger, R., Kwon, P. and Dharan, C.K.H. (1994), "High speed centrifugal casting of metal matrix composites", *Proceedings of the 5th International Symposium on Transport Phonomena and Dynamics of Rotating Machinery*, Maui, HI, USA.

Bharti, I., Gupta, N. and Gupta, K.M. (2013), "Novel Applications of Functionally Graded Nano, Optoelectronic and Thermoelectric Materials", *Int. J. Mater. Mech. Manuf.*, **1**(3), 221-224.
DOI: 10.7763/IJMMM.2013.V1.47

Ebrahimi, F. and Barati, M.R. (2017a), "Thermal-induced nonlocal vibration characteristics of hetero-geneous beams", *Adv. Mater. Res., Int. J.*, **6**(2), 93-128. DOI: 10.12989/amr.2017.6.2.093

Ebrahimi, F. and Barati, M.R. (2017b), "Buckling analysis of nonlocal strain gradient axially functionally graded nanobeams resting on variable elastic medium", *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, 095440621771351.
DOI: 10.1177/0954406217713518

Ebrahimi, F. and Barati, M.R. (2017c), "Through-the-length temperature distribution effects on thermal vibration analysis of nonlocal strain-gradient axially graded nanobeams subjected to nonuniform magnetic field", *J. Thermal Stress.*, **40**(5), 548-563. DOI: 10.1080/01495739.2016.1254076

Ebrahimi, F. and Barati, M.R. (2017d), "Longitudinal varying elastic foundation effects on vibration behavior of axially graded nanobeams via nonlocal strain gradient elasticity theory", *Mech. Adv. Mater. Struct.*, **6494**, 1-11. DOI: 10.1080/15376494.2017.1329467

Ebrahimi, F. and Jafari, A. (2017), "Investigating vibration behavior of smart imperfect functionally graded beam subjected to magnetic-electric fields based on refined shear deformation theory", *Adv. Nano Res., Int. J.*, **5**(4), 281-301. DOI:10.12989/anr.2017.5.4.281

Ebrahimi, F., Mahmoodi, F. and Barati, M.R. (2017), "Thermo-mechanical vibration analysis of functionally graded micro/nanoscale beams with porosities based on modified couple stress theory", *Adv. Mater. Res., Int. J.*, **6**(3), 279-301. DOI:10.12989/amr.2017.6.3.279

Ebrahimi, F., Barati, M.R. and Zenkour, A.M. (2018), "A new nonlocal elasticity theory with graded nonlocality for thermo-mechanical vibration of FG nanobeams via a nonlocal third-order shear deformation theory", *Mech. Adv. Mater. Struct.*, **25**(6), 512-522. DOI: 10.1080/15376494.2017.1285458

Elishakoff, I. and Guédé, Z. (2004), "Analytical polynomial solutions for vibrating axially graded beams", *Mech. Adv. Mater. Struct.*, **11**(6 II), 517-533. DOI: 10.1080/15376490490452669

Elishakoff, I. and Johnson, V. (2005), "Apparently the first closed-form solution of vibrating inhomogeneous beam with a tip mass", *J. Sound Vib.*, **286**(4-5), 1057-1066.
DOI: 10.1016/j.jsv.2005.01.050

Eringen, A.C. (1976), "Part III – Nonlocal Polar Field Theories", *Continuum Phys.*, 205-267.

Eringen, A.C. (1983), "On differential equations of nonlocal elasticity and solutions of screw dislocation and

surface waves", *J. Appl. Phys.*, **54**(9), 4703-4710. DOI: 10.1063/1.332803

Ferreira, A.J.M., Batra, R.C., Roque, C.M.C., Qian, L.F. and Jorge, R.M.N. (2006), "Natural frequencies of functionally graded plates by a meshless method", *Compos. Struct.*, **75**(1-4), 593-600. DOI: 10.1016/j.compstruct.2006.04.018

Fukui, Y. (1990), "Fundamental investigation of functionally gradient material manufacturing system using centrifugal force", *Transact. Japan Soc. Mech. Eng. Series C*, **56**(521), 67-70. DOI: 10.1299/kikaic.56.67

Hosseini-Hashemi, S. and Nazemnezhad, R. (2013), "An analytical study on the nonlinear free vibration of functionally graded nanobeams incorporating surface effects", *Compos. Part B*: *Eng.*, **52**, 199-206. DOI: 10.1016/j.compositesb.2013.04.023

Hu, Y.G., Liew, K.M., Wang, Q., He, X.Q. and Yakobson, B.I. (2008), "Nonlocal shell model for elastic wave propagation in single- and double-walled carbon nanotubes", *J. Mech. Phys. Solids*, **56**(12), 3475-3485. DOI: 10.1016/j.jmps.2008.08.010

Huang, Y. and Li, X.F. (2010), "A new approach for free vibration of axially functionally graded beams with non-uniform cross-section", *J. Sound Vib.*, **329**(11), 2291-2303. DOI: 10.1016/j.jsv.2009.12.029

Huang, Y., Yang, L.E. and Luo, Q.Z. (2013), "Free vibration of axially functionally graded Timoshenko beams with non-uniform cross-section", *Compos. Part B*: *Eng.*, **45**(1), 1493-1498. DOI: 10.1016/j.compositesb.2012.09.015

Janghorban, M. and Zare, A. (2011), "Free vibration analysis of functionally graded carbon nanotubes with variable thickness by differential quadrature method", *Phys. E*: *Low-Dimen. Syst. Nanostruct.*, **43**(9), 1602-1604. DOI: 10.1016/j.physe.2011.05.002

Lambros, J., Narayanaswamy, A., Santare, M.H. and Anlas, G. (1999), "Manufacture and testing of a functionally graded material", *J. Eng. Mater. Technol.*, **121**(October), 488–493. DOI: 10.1115/1.2812406

Li, X., Li, L., Hu, Y., Ding, Z. and Deng, W. (2017), "Bending, buckling and vibration of axially functionally graded beams based on nonlocal strain gradient theory", *Compos. Struct.*, **165**, 250-265. DOI: 10.1016/j.compstruct.2017.01.032

Lu, P., Lee, H.P., Lu, C. and Zhang, P.Q. (2007), "Application of nonlocal beam models for carbon nanotubes", *Int. J. Solids Struct.*, **44**(16), 5289-5300. DOI: 10.1016/j.ijsolstr.2006.12.034

Moya, J.S. (1995), "Layered ceramics", *Adv. Mater.*, **7**(2), 185-189. DOI: 10.1002/adma.19950070219

Nejad, M.Z. and Hadi, A. (2016), "Non-local analysis of free vibration of bi-directional functionally graded Euler-Bernoulli nano-beams", *Int. J. Eng. Sci.*, **105**, 1-11. DOI: 10.1016/j.ijengsci.2016.04.011

Peddieson, J., Buchanan, G.R. and McNitt, R.P. (2003), "Application of nonlocal continuum models to nanotechnology", *Int. J. Eng. Sci.*, **41**(3-5), 305-312. DOI: 10.1016/S0020-7225(02)00210-0

Qian, L.F. and Batra, R.C. (2005), "Design of bidirectional functionally graded plate for optimal natural frequencies", *J. Sound Vib.*, **280**(1-2), 415-424. DOI: 10.1016/j.jsv.2004.01.042

Qian, L.F., Batra, R.C. and Chen, L.M. (2004), "Static and dynamic deformations of thick functionally graded elastic plates by using higher-order shear and normal deformable plate theory and meshless local Petrov–Galerkin method", *Compos. Part B*: *Eng.*, **35**(6-8), 685-697. DOI: 10.1016/j.compositesb.2004.02.004

Robinson, M.T.A. and Adali, S. (2016), "Variational solution for buckling of nonlocal carbon nanotubes under uniformly and triangularly distributed axial loads", *Compos. Struct.* DOI: 10.1016/j.compstruct.2016.01.026

Sedighi, H.M., Daneshmand, F. and Abadyan, M. (2015), "Dynamic instability analysis of electrostatic functionally graded doubly-clamped nano-actuators", *Compos. Struct.*, **124**, 55-64. DOI: 10.1016/j.compstruct.2015.01.004

Shahba, A., Attarnejad, R., Marvi, M.T. and Hajilar, S. (2011), "Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams with classical and non-classical boundary conditions", *Compos. Part B*: *Eng.*, **42**(4), 801-808. DOI: 10.1016/j.compositesb.2011.01.017

Shafiei, N., Kazemi, M. and Ghadiri, M. (2016), "Comparison of modeling of the rotating tapered axially functionally graded Timoshenko and Euler-Bernoulli microbeams", *Phys. E*: *Low-Dimen. Syst. Nanostruct.*, **83**, 74-87. DOI: 10.1016/j.physe.2016.04.011

Shafiei, N., Mirjavadi, S.S., Afshari, B.M., Rabby, S. and Hamouda, A.M.S. (2017), "Nonlinear thermal

buckling of axially functionally graded micro and nanobeams", *Compos. Struct.*, **168**, 428-439.
  DOI: 10.1016/j.compstruct.2017.02.048

Şimşek, M. (2010), "Fundamental frequency analysis of functionally graded beams by using different
  higher-order beam theories", *Nucl. Eng. Des.*, **240**(4), 697-705. DOI: 10.1016/j.nucengdes.2009.12.013

Şimşek, M. (2012), "Nonlocal effects in the free longitudinal vibration of axially functionally graded tapered
  nanorods", *Computat. Mater. Sci.*, **61**, 257-265. DOI: 10.1016/j.commatsci.2012.04.001

Sudak, L.J. (2003), "Column buckling of multiwalled carbon nanotubes using nonlocal continuum
  mechanics", *J. Appl. Phys.*, **94**(11), 7281-7287. DOI: 10.1063/1.1625437

Uymaz, B. and Aydogdu, M. (2007), "Three-dimensional vibration analyses of functionally graded plates
  under various boundary conditions", *J. Reinf. Plast. Compos.*, **26**(18), 1847-1863.
  DOI: 10.1177/0731684407081351

Vel, S.S. and Batra, R.C. (2004), "Three-dimensional exact solution for the vibration of functionally graded
  rectangular plates", *J. Sound Vib.*, **272**(3-5), 703-730. DOI: 10.1016/S0022-460X(03)00412-7

Wang, Q. (2005), "Wave propagation in carbon nanotubes via nonlocal continuum mechanics", *J. Appl.
  Phys.*, **98**(12), 124301. DOI: 10.1063/1.2141648

Wu, L., Wang, Q.S. and Elishakoff, I. (2005), "Semi-inverse method for axially functionally graded beams
  with an anti-symmetric vibration mode", *J. Sound Vib.*, **284**(3-5), 1190-1202.
  DOI: 10.1016/j.jsv.2004.08.038

Yamanouchi, M., Koizumi, M., Hirai, T. and Shiota, I. (1990), "On the design of functionally gradient
  materials", *Proceedings of the 1st International Symposium on Functionally Gradient Materials*, pp. 5-10.

Yang, J.E., Park, W.H., Kim, C.J., Kim, Z.H. and Jo, M.H. (2008), "Axially graded heteroepitaxy and
  Raman spectroscopic characterizations of Si1-xGex nanowires", *Appl. Phys. Lett.*, **92**(26), 3-6.
  DOI: 10.1063/1.2939564

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