Advances in Nano Research, *Vol. 6 No. 2 (2018) 113-133*
DOI: https://doi.org/10.12989/anr.2018.6.2.113

Modeling the size effect on vibration characteristics of functionally graded piezoelectric nanobeams based on Reddy's shear deformation beam theory

Farzad Ebrahimi [*1] and Ramin Ebrahimi Fardshad [2]

[1] *Department of Mechanical Engineering, Faculty of Engineering, Imam Khomeini International University, Qazvin, Iran*
[2] *Faculty of Industrial and Mechanical Engineering, Qazvin Branch, Islamic Azad University, Qazvin, Iran*

(*Received June 23, 2017, Revised September 25, 2017, Accepted February 2, 2018*)

Abstract. In this work, free vibration characteristics of functionally graded piezoelectric (FGP) nanobeams based on third order parabolic shear deformation beam theory are studied by presenting a Navier type solution as the first attempt. Electro-mechanical properties of FGP nanobeam are supposed to change continuously throughout the thickness based on power-law model. To capture the small size effects, Eringen's nonlocal elasticity theory is adopted. Using Hamilton's principle, the nonlocal governing equations for third order shear deformable piezoelectric FG nanobeams are obtained and they are solved applying analytical solution. By presenting some numerical results, it is demonstrated that the suggested model presents accurate frequency results of the FGP nanobeams. The influences of several parameters including, external electric voltage, power-law exponent, nonlocal parameter and mode number on the natural frequencies of the size-dependent FGP nanobeams is discussed in detail.

Keywords: functionally graded piezoelectric nanobeam; free vibration; nonlocal elasticity theory; Reddy beam theory

1. Introduction

Piezoelectric materials can couple electrical and mechanical energy through linking electric signals to material stress ans strain and hence when piezoelectric materials are exposed to an external electrical voltage, it gets deformed. So, the application of piezoelectric materials for the vibration reduction and shape control is fast becoming an essential tool in the design of smart structures and systems. The novel spatial composite materials called functionally graded materials (FGMs) are consist of two or more material constituents such as a pair of ceramic and metal in which their volume fractions are supposed to change continuously throughout the desired directions. The FGM constituents provide various advantageous features, for example, the ceramic constituents are capable to endure severe temperature environments due to their better thermal resistance characteristics, whereas the metal constituents provide better mechanical performance and diminishes the possibility of disastrous fracture.

∗Corresponding author, Ph.D., Professor, E-mail: febrahimy@eng.ikiu.ac.ir

http://www.techno-press.org/?journal=anr&subpage=5 ISSN: 2287-237X (Print), 2287-2388 (Online)

Micro and nano electro-mechanical systems (MEMS/NEMS) are composed of several structural elements including nanoscale beams and plates which have excellent mechanical, chemical, and electronic properties. So, nano scale structures achieved intense interest by researchers based on molecular dynamics and continuum mechanics. The trouble in utilization of classical theory is that the classical continuum mechanics theory is impotent to cpture the small size effects in structures at micro and nano scales and finally over predicts the responses of these micro/nano scale structures. Molecular dynamic simulations (MD) is an alternative approach which can capture the size effects. But, when the molecular dynamic simulation applies to the nanostructures, it seems to be computationally very ponderous. Hence, Eringen's nonlocal elasticity theory is presented to overcome these problems and hence, small size effects in modeling of nanostructures is captured with excellent accuracy. The nonlocal elasticity theory of Eringen supposes that the stress state at a desired point is a function of the strain at all neighbor points of the body. In recent years extensive studies is performed to analyze the static and dynamic behavior of size-dependent FG beams. Recently, Eltaher *et al*. (2012, 2013a) presented a finite element analysis for free vibration of FG nanobeams using nonlocal EBT. They also exploited the static and stability responses of FG nanobeams based on nonlocal continuum theory (Eltaher *et al*. 2013b). More recently, using nonlocal TBT and EBT, Simsek and Yurtcu (2013) investigated bending and buckling of FG nanobeam by analytical method. Sharabiani and Yazdi (2013) investigated nonlinear free vibration of FG nanobeams within the framework of Euler–Bernoulli beam model including the von Kármán geometric nonlinearity. Forced vibration analysis of FG nanobeams based on the nonlocal elasticity theory and using Navier method for various shear deformation theories studied by Uymaz (2013). Rahmani and Pedram (2014) analyzed the size effects on vibration of FG nanobeams based on nonlocal TBT. Nonlinear free vibration of FG nanobeams with fixed ends, i.e., simply supported-simply supported (SS) and simply supported-clamped (SC), using the nonlocal elasticity within the frame work of EBT with von kármán type nonlinearity is studied by Nazemnezhad and Hosseini-Hashemi (2014). Also, recently Hosseini-Hashemi *et al*. (2014) investigated free vibration of FG nanobeams with consideration surface effects and piezoelectric field using nonlocal elasticity theory. Most recently Ebrahimi *et al*. (Ebrahimi *et al*. 2015, Ebrahimi and Salari 2015) examined the applicability of differential transformation method in investigations on vibrational characteristics of FG size-dependent nanobeams. In another work, Ebrahimi and Salari (2015a) presented a semi-analytical method for vibrational and buckling analysis of FG nanobeams considering the position of neutral axis. An exact solution for the nonlinear forced vibration of FG nanobeams in thermal environment based on surface elasticity theory in presented by Ansari *et al*. (2015). Recently, Rahmani and Jandaghian (2015) presented Buckling analysis of FG nanobeams based on a nonlocal third-order shear deformation theory. Most recently Li and Hu (2017a) investigated torsional vibration of bi-directional FG nanotubes based on nonlocal elasticity theory. They (Li and Hu 2017b) also presented a post-buckling analysis of FG nanobeams incorporating nonlocal stress and microstructure-dependent strain gradient effects. In other work Eringen's nonlocal integral model is presented to model the twisting statics of FG nanotubes by Zhu and Li (2017).

Moreover, several investigations are carried out to study responses of FGP material beams. The problem of a FGP cantilever beam exposed to various loadings is studied by Shi and Chen (2004). They characterized the piezoelectric beam by continuously graded properties for one elastic parameter and the material density. Bending and free vibration responses of monomorph, bimorph, and multimorph actuators made of FGP materials under a combined thermal-electro-mechanical load based upon Timoshenko beam model studied by Yang and Xiang (2007). Doroushi *et al*.

(2011) investigated the free and forced vibration characteristics of an FGPM beam subjected to thermo-electro-mechanical loads using the higher-order shear deformation beam theory. Kiani *et al*. (2011) analysed buckling behavior of FGM beams with or without surface-bonded piezoelectric layers subjected to both thermal loading and constant voltage. Komijani *et al*. (2013) studied free vibration of FGP beams with rectangular cross sections under in-plane thermal and electrical excitations in pre/post-buckling regimes. Lezgy-Nazargah *et al*. (2013) suggested an efficient three-nodded beam element model for static, free vibration and dynamic response of functionally graded piezoelectric material beams. Also, Lezgy-Nazargah (2016) investigated presented a three-dimensional Peano series solution for the vibration of FGP laminates in cylindrical bending. He (Lezgy-Nazargah 2015) also investigated the cylindrical bending of continuously non-homogenous piezoelectric laminated plates with arbitrary gradient composition via an exact state-space solution. Large amplitude free flexural vibration of shear deformable FG beams with surface-bonded piezoelectric layers subjected to thermopiezoelectric loadings with random material properties presented by Shegokar and Lal (2014). Li *et al*. (2014) developed a size-dependent FGP beam model using the modified strain gradient theory and Timoshenko beam theory. Therefore it could be noted that the main deficiency of above-mentioned studies is that the small size effects is not considered in these works. In other work, Filippi (2015) analyzed static behavior of FGM beams by various theories and finite elements. To capture the size effect, recently a parametric study is performed to explore the influences of size-dependent shear deformation on static bending, buckling and free vibration behavior of microbeams based on modified couple stress classical and first shear deformation beam models by Dehrouyeh-Semnani and Nikkhah-Bahrami (2015). They indicated that the influence of size-dependent shear deformation on mechanical behavior of the microbeams has an ascending trend with respect to dimensionless material length scale parameter.

Therefore, it is clear that a work to study vibrational responses of FGP nanobeams using a parabolic shear deformation beam theory is not yet published. It can be seen that most of recent works for studing influences of piezoelectric materials on mechanical behavior of FG nanobeams have done based on Euler-Bernoulli (EBT) and Timoshenko beam (TBT) theories. It is well known that Euler-Bernoulli beam model fails to capture the influences of shear deformations and hence the buckling loads and natural frequencies of nanobeams are overestimated. Timoshenko beam model has the potencial to capture the influences of shear deformations, but a shear correction factor is required to perfect demonstration of the deformation strain energy. Several higher-order shear deformation theories which are needless of shear correction factors are introduced such as the parabolic shear deformation theory proposed by Reddy (2007), the generalized beam theory proposed by Aydogdu (2009), sinusoidal shear deformation theory of Touratier (1991) and hyperbolic shear deformation presented by Soldatos (1992).

The present study deals with the free vibration analysis of simply supported FGP piezoelectric nanobeams based on third order shear deformation beam theory. The electro-mechanical material properties of the beam is supposed to be graded in the thickness direction according to the power law distribution. Applying non-classical higher order beam model and Eringen’s nonlocal elasticity theory, the small size effect is captured. Nonlocal governing equations for the free vibration of a higher order FG nanobeam have been derived via Hamilton’s principle. Derived equations are solved using Navier type method and several numerical examples are presented investigating the effects of external electric voltage, power-law index, mode number and small scale parameter on vibration characteristics of size-dependent FGP piezoelectric nanobeams.

2. Theoretical formulations

2.1 The material properties of FGP nanobeams

Assume a functionally graded nanobeam composed of PZT-4 and PZT-5H piezoelectric materials exposed to an electric potential Φ (*x*, *z*, *t*), with length *L* and uniform thickness *h*, as shown in Fig. 1. The effective material properties of the FGPM nanobeam are supposed to change continuously in the *z*-axis direction (thickness direction) based on the power-law model. So, the effective material properties, *P*, can be stated in the following form (Komijani *et al*. 2013)

$$P = P_2 V_2 + P_1 V_1 \tag{1}$$

in which P_1 and P_2 denote the material properties of the bottom and higher surfaces, respectively. Also, V_1 and V_2 are the corresponding volume fractions related by

$$V_2 = \left(\frac{z}{h} + \frac{1}{2}\right)^p, \quad V_1 = 1 - V_2 \tag{2}$$

Therefore, according to Eqs. (1) and (2), the effective electro-mechanical material properties of the FGP beam is defined as

$$P(z) = \left(P_2 - P_1\right)\left(\frac{z}{h} + \frac{1}{2}\right)^p + P_1 \tag{3}$$

where *p* is power-law exponent which is t non-negative and estimates the material distribution through the thickness of the nanobeam and *z* is the distance from the mid-plane of the graded piezoelectric beam. It must be noted that, the top surface at $z = +h/2$ of FGP nanobeam is assumed PZT-4 rich, whereas the bottom surface ($z = -h/2$) is PZT-5H rich.

2.2 Nonlocal elasticity theory for the piezoelectric materials

Contrary to the constitutive equation of classical elasticity theory, Eringen’s nonlocal theory

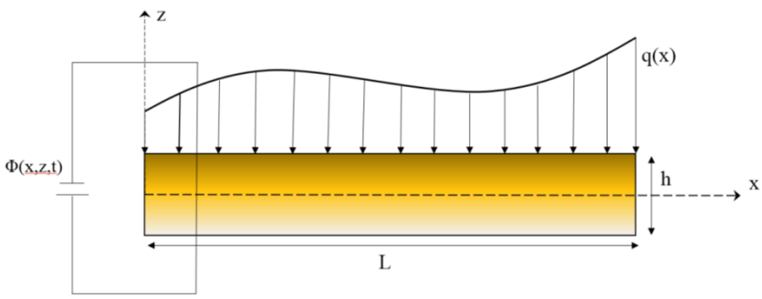


Fig. 1 Configuration of a functionally graded piezoelectric nanobeam

(Eringen 1972, 1983, Eringen *et al*. 1972) notes that the stress state at a point inside a body is regarded to be function of strains of all points in the neighbor regions. For a nonlocal homogeneous piezoelectric solid the basic equations with zero body force may be defined as

$$\sigma_{ij} = \int_V \alpha\left(\left|x' - x\right|, \tau\right)\left[C_{ijkl}\varepsilon_{kl}(x') - e_{kij}E_k(x')\right]dV(x') \tag{4a}$$

$$D_i = \int_V \alpha\left(\left|x' - x\right|, \tau\right)\left[e_{ikl}\varepsilon_{kl}(x') + k_{ik}E_k(x')\right]dV(x') \tag{4b}$$

where σ_{ij}, ε_{ij}, D_i and E_i denote the stress, strain, electric displacement and electric field components, respectively; C_{ijkl}, e_{kij} and k_{ik} are elastic, piezoelectric and dielectric constant, respectively; α ($|x' - x|$, τ) is the nonlocal kernel function and $|x' - x|$ is the Euclidean distance. $\tau = e_0 a/l$ is defined as scale coefficient, where e_0 is a material constant which is determined experimentally or approximated by matching the dispersion curves of plane waves with those of atomic lattice dynamics; and a and l are the internal and external characteristic length of the nanostructures, respectively. Finally it is possible to represent the integral constitutive relations given by Eq. (4) in an equivalent differential form as

$$\sigma_{ij} - (e_0\, a)^2 \nabla^2 \sigma_{ij} = C_{ijkl}\varepsilon_{kl} - e_{kij}E_k \tag{5a}$$

$$D_i - (e_0\, a)^2 \nabla^2 D_i = e_{ikl}\varepsilon_{kl} + k_{ik}E_k \tag{5b}$$

where ∇^2 is the Laplacian operator and $e_0 a$ is the nonlocal parameter revealing the size influence on the response of nanostructures.

2.3 Nonlocal FG piezoelectric nanobeam model

Based on parabolic third order beam theory, the displacement field at any point of the beam are supposed to be in the form

$$u_x(x, z) = u(x) + z\psi(x) - \alpha z^3 (\psi + \frac{\partial w}{\partial x}) \tag{6a}$$

$$u_z(x, z) = w(x) \tag{6b}$$

in which u and w are displacement components in the mid-plane along the coordinates x and z, respectively, while ψ denotes the total bending rotation of the cross-section.

To satisfy Maxwell's equation in the quasi-static approximation, the distribution of electric potential along the thickness direction is supposed to change as a combination of a cosine and linear variation as follows

$$\Phi(x, z, t) = -\cos(\xi z)\phi(x, t) + \frac{2z}{h}V \tag{7}$$

where $\xi = \pi/h$. Also, V is the initial external electric voltage applied to the FGP nanobeam; and ϕ

(x, t) is the spatial function of the electric potential in the x-direction. Considering strain–displacement relationships on the basis of parabolic beam theory, the non-zero strains can be stated as

$$\varepsilon_{xx} = \varepsilon_{xx}{}^{(0)} + z\varepsilon_{xx}{}^{(1)} + z^3\varepsilon_{xx}{}^{(3)} \tag{8}$$

$$\gamma_{xz} = \gamma_{xz}{}^{(0)} + z^2\gamma_{xz}{}^{(2)} \tag{9}$$

where

$$\varepsilon_{xx}{}^{(0)} = \frac{\partial u}{\partial x},\ \varepsilon_{xx}{}^{(1)} = \frac{\partial \psi}{\partial x},\ \varepsilon_{xx}{}^{(3)} = -\alpha(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2}) \tag{10}$$

$$\gamma_{xz}{}^{(0)} = \frac{\partial w}{\partial x} + \psi,\ \gamma_{xz}{}^{(2)} = -\beta(\frac{\partial w}{\partial x} + \psi) \tag{11}$$

and $\beta = \frac{4}{h^2}$. According to the defined electric potential in Eq. (7), the non-zero components of electric field (E_x, E_z) can be obtained as

$$E_x = -\Phi_{,x} = \cos(\xi z)\frac{\partial \phi}{\partial x},\quad E_z = -\Phi_{,z} = -\xi\sin(\xi z)\phi - \frac{2V_E}{h} \tag{12}$$

The Hamilton's principle can be stated in the following form to obtain the governing equations of motion

$$\int_0^t \delta(\Pi_S - \Pi_K + \Pi_W)dt = 0 \tag{13}$$

where Π_S is strain energy, Π_K is kinetic energy and Π_W is work done by external applied forces. The first variation of strain energy Π_S can be calculated as

$$\delta\Pi_S = \int_0^L \int_{-h/2}^{h/2} \left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{xz}\delta\gamma_{xz} - D_x\delta E_x - D_z\delta E_z\right)dzdx \tag{14}$$

Substituting Eqs. (8) and (9) into Eq. (14) yields

$$\begin{aligned}\delta\Pi_S &== \int_0^L (N\delta\varepsilon_{xx}{}^{(0)} + M\delta\varepsilon_{xx}{}^{(1)} + P\delta\varepsilon_{xx}{}^{(3)} + Q\delta\gamma_{xz}{}^{(0)} + R\delta\gamma_{xz}{}^{(2)})dx \\ &+ \int_0^L \int_{-h/2}^{h/2} \left(-D_x\cos(\beta z)\delta\left(\frac{\partial \phi}{\partial x}\right) + D_z\beta\sin(\beta z)\delta\phi\right)dzdx\end{aligned} \tag{15}$$

in which N, M and Q are the axial force, bending moment and shear force resultants, respectively. Relations between the stress resultants and stress component used in Eq. (15) are defined as

$$\begin{aligned}&N = \int_A \sigma_{xx}\,dA,\ M = \int_A \sigma_{xx} z\,dA,\ P = \int_A \sigma_{xx} z^3 dA \\ &Q = \int_A \sigma_{xz}\,dA,\ R = \int_A \sigma_{xz} z^2\,dA\end{aligned} \tag{16}$$

The kinetic energy Π_K for graded piezoelectric nanobeam is formulated as

$$\Pi_K = \frac{1}{2}\int_0^L \int_{-h/2}^{h/2} \rho\left((\frac{\partial u_x}{\partial t})^2 + (\frac{\partial u_z}{\partial t})^2 \right) dzdx \tag{17}$$

where ρ is the mass density. The first variation of the kinetic energy is presented as

$$\begin{aligned}\Pi_K = \int_0^L & I_0 (\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t}) + I_1(\frac{\partial u}{\partial t}\frac{\partial \delta \psi}{\partial t} + \frac{\partial \psi}{\partial t}\frac{\partial \delta u}{\partial t}) + I_2 \frac{\partial \psi}{\partial t}\frac{\partial \delta \psi}{\partial t} \\ & + \alpha[-I_3 \frac{\partial u}{\partial t}(\frac{\partial^2 \delta w}{\partial x \partial t} + \frac{\partial \delta \psi}{\partial t}) - I_3 \frac{\partial \delta u}{\partial t}(\frac{\partial^2 w}{\partial x \partial t} + \frac{\partial \psi}{\partial t}) - I_4 \frac{\partial \psi}{\partial t}(\frac{\partial \delta \psi}{\partial t} + \frac{\partial^2 \delta w}{\partial x \partial t}) \\ & - I_4 \frac{\partial \delta \psi}{\partial t}(\frac{\partial \psi}{\partial t} + \frac{\partial^2 w}{\partial x \partial t}) + \alpha I_6 (\frac{\partial \psi}{\partial t} + \frac{\partial^2 w}{\partial x \partial t})(\frac{\partial \delta \psi}{\partial t} + \frac{\partial^2 \delta w}{\partial x \partial t})] dAdx\end{aligned} \tag{18}$$

In which I_0, I_1, I_2, I_3, I_4 and I_6 are mass inertia and defined as

$$(I_0, I_1, I_2, I_3, I_4, I_6) = \int_A (1, z, z^2, z^3, z^4, z^6)\rho dA \tag{19}$$

It is noticed from Eq. (19), for homogeneous nanobeams, $I_2 = I_3 = 0$. The work done due to external electric voltage, Π_W can be written in the form

$$\Pi_w = \int_0^L (N_E \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + q\delta w + f\delta u - N\delta \varepsilon_{xx}{}^{(0)} - M \frac{\partial \delta \psi}{\partial x} + \alpha P \frac{\partial^2 \delta w}{\partial x^2} - Q\delta \gamma_{xz}{}^{(0)}) dx \tag{20}$$

where $M = M - \alpha P$, $Q = Q - \beta R$ and $q(x)$ and $f(x)$ are the transverse and axial distributed loads and N_E is normal forced due to external electric voltage (V) which is defined as

$$N_E = -\int_{-h/2}^{h/2} e_{31} \frac{2V}{h} dz \tag{21}$$

For a FGPM nanobeam exposed to electro-mechanical loading in the one dimensional case, the nonlocal constitutive relations (5a) and (5b) may be rewritten as

$$\sigma_{xx} - (e_0 a)^2 \frac{\partial^2 \sigma_{xx}}{\partial x^2} = c_{11}\varepsilon_{xx} - e_{31}E_z \tag{22}$$

$$\sigma_{xz} - (e_0 a)^2 \frac{\partial^2 \sigma_{xz}}{\partial x^2} = c_{55}\gamma_{xz} - e_{15}E_x \tag{23}$$

$$D_x - (e_0 a)^2 \frac{\partial^2 D_x}{\partial x^2} = e_{15}\gamma_{xz} + k_{11}E_x \tag{24}$$

$$D_z - (e_0 a)^2 \frac{\partial^2 D_z}{\partial x^2} = e_{31}\varepsilon_{xx} + k_{33}E_z \tag{25}$$

Inserting Eqs. (15), (17) and (20) in Eq. (13) and integrating by parts, and gathering the coefficients of δu, δw, $\delta\psi$ and $\delta\phi$, the following governing equations are obtained

$$\frac{\partial N}{\partial x} + f - I_0 \frac{\partial^2 u}{\partial t^2} - \hat{I}_1 \frac{\partial^2 \psi}{\partial t^2} + \alpha I_3 \frac{\partial^3 w}{\partial x \partial t^2} = 0 \tag{26}$$

$$\frac{\partial M}{\partial N} - Q - \hat{I}_1 \frac{\partial^2 u}{\partial t^2} - \hat{I}_2 \frac{\partial^2 \psi}{\partial t^2} + \alpha \hat{I}_4 \left(\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^3 \psi}{\partial x \partial t^2}\right) = 0 \tag{27}$$

$$\frac{\partial Q}{\partial x} + q - N_E \frac{\partial^2 w}{\partial x^2} - \alpha \frac{\partial^2 P}{\partial x^2} - I_0 \frac{\partial^2 w}{\partial t^2} - \alpha I_3 \frac{\partial^3 u}{\partial x \partial t^2} - \alpha I_4 \frac{\partial^3 \psi}{\partial x \partial t^2} + \alpha^2 I_6 \left(\frac{\partial^2 \psi}{\partial x \partial t^2} + \frac{\partial^4 w}{\partial x^2 \partial t^2}\right) = 0 \tag{28}$$

$$\int_{-h/2}^{h/2} \left(\cos(\beta z) \frac{\partial D_x}{\partial x} + \beta \sin(\beta z) D_z \right) dz = 0 \tag{29}$$

By integrating Eqs. (22)-(25), over the beam's cross-section area, the force-strain and the moment-strain of the nonlocal third order Reddy FGP beam theory can be obtained as follows

$$N - \mu \frac{\partial^2 N}{\partial x^2} = A_{xx} \frac{\partial u}{\partial x} + B_{xx} \frac{\partial \psi}{\partial x} - \alpha E_{xx} \left(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + A_{31}^e \phi - N_E \tag{30}$$

$$M - \mu \frac{\partial^2 M}{\partial x^2} = B_{xx} \frac{\partial u}{\partial x} + D_{xx} \frac{\partial \psi}{\partial x} - \alpha F_{xx} \left(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + E_{31}\phi \tag{31}$$

$$P - \mu \frac{\partial^2 P}{\partial x^2} = E_{xx} \frac{\partial u}{\partial x} + F_{xx} \frac{\partial \psi}{\partial x} - \alpha H_{xx} \left(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + F_{31}\phi \tag{32}$$

$$Q - \mu \frac{\partial^2 Q}{\partial x^2} = (A_{xz} - \beta D_{xz})\left(\frac{\partial w}{\partial x} + \psi\right) - E_{15} \frac{\partial \phi}{\partial x} \tag{33}$$

$$R - \mu \frac{\partial^2 R}{\partial x^2} = (D_{xz} - \beta F_{xz})\left(\frac{\partial w}{\partial x} + \psi\right) - F_{15} \frac{\partial \phi}{\partial x} \tag{34}$$

$$\int_{-h/2}^{h/2} \left\{ D_x - \mu \frac{\partial^2 D_x}{\partial x^2} \right\} \cos(\xi z) dz = (E_{15} - \beta F_{15})\left(\frac{\partial w}{\partial x} + \psi\right) + F_{11} \frac{\partial \phi}{\partial x} \tag{35}$$

$$\int_{-h/2}^{h/2} \left\{ D_z - \mu \frac{\partial^2 D_z}{\partial x^2} \right\} \xi \sin(\xi z) dz = A_{31}^e \frac{\partial u}{\partial x} + (E_{31} - \alpha F_{31}) \frac{\partial \psi}{\partial x} - \alpha F_{31} \frac{\partial^2 w}{\partial x^2} - F_{33}\phi \tag{36}$$

where $\mu = (e_0 a)^2$ and quantities used in above equations are defined as

$$\left\{A_{xx}, B_{xx}, D_{xx}, E_{xx}, F_{xx}, H_{xx}\right\} = \int_{-h/2}^{h/2} c_{11} \left\{1, z, z^2, z^3, z^4, z^6\right\} dz \tag{37}$$

$$\left\{A_{xz}, D_{xz}, F_{xz}\right\} = \int_{-h/2}^{h/2} c_{55} \left\{1, z^2, z^4\right\} dz \tag{38}$$

$$\left\{A_{31}^e, E_{31}, F_{31}\right\} = \int_{-h/2}^{h/2} e_{31} \left\{ \xi \sin(\xi z), z\xi \sin(\xi z), z^3 \xi \sin(\xi z)\right\} dz \tag{39}$$

$$\left\{E_{15}, F_{15}\right\} = \int_{-h/2}^{h/2} e_{15} \left\{ \cos(\xi z), z^2 \cos(\xi z)\right\} dz \tag{40}$$

$$\left\{F_{11}, F_{33}\right\} = \int_{-h/2}^{h/2} \left\{k_{11} \cos^2(\xi z), k_{33} \xi^2 \sin^2(\xi z)\right\} dz \tag{41}$$

The explicit relation of the nonlocal normal force can be derived by substituting for the second derivative of N from Eq. (26) into Eq. (30) as follows

$$\begin{aligned} N_x = {} & A_{xx} \frac{\partial u}{\partial x} + K_{xx} \frac{\partial \psi}{\partial x} - \alpha E_{xx} \frac{\partial^2 w}{\partial x^2} + A_{31}^e \phi - N_E \\ & + \mu\left(-\frac{\partial f}{\partial x} + I_0 \frac{\partial^3 u}{\partial x \partial t^2} + \hat{I}_1 \frac{\partial^3 \psi}{\partial x \partial t^2} - \alpha I_3 \frac{\partial^4 w}{\partial x^2 \partial t^2}\right) \end{aligned} \tag{42}$$

Omitting Q from Eqs. (27) and (28), we obtain the following equation

$$\frac{\partial^2 M}{\partial x^2} = -\alpha \frac{\partial^2 P}{\partial x^2} - q + N_E \frac{\partial^2 w}{\partial x^2} + I_0 \frac{\partial^2 w}{\partial t^2} + I_1 \frac{\partial^3 u}{\partial x \partial t^2} + I_2 \frac{\partial^3 \psi}{\partial x \partial t^2} - \alpha I_4 \left(\frac{\partial^3 \psi}{\partial x \partial t^2} + \frac{\partial^4 w}{\partial x^2 \partial t^2}\right) \tag{43}$$

Also the explicit relation of the nonlocal bending moment can be derived by substituting for the second derivative of M from Eq. (27) into Eq. (31) and using Eqs. (31) and (32) as follows

$$\begin{aligned} M = {} & K_{xx} \frac{\partial u}{\partial x} + I_{xx} \frac{\partial \psi}{\partial x} - \alpha J_{xx} \left(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + (E_{31} - \alpha F_{31})\phi + \mu\left(-\alpha \frac{\partial^2 P}{\partial x^2} - q + \frac{\partial}{\partial x}\left(N_E \frac{\partial w}{\partial x}\right)\right. \\ & \left. + I_0 \frac{\partial^2 w}{\partial t^2} + I_1 \frac{\partial^3 \psi}{\partial x \partial t^2} + I_2 \frac{\partial^3 \psi}{\partial x \partial t^2} - \alpha I_4 \left(\frac{\partial^3 \psi}{\partial x \partial t^2} + \frac{\partial^4 w}{\partial x^2 \partial t^2}\right)\right) \end{aligned} \tag{44}$$

where

$$K_{xx} = B_{xx} - \alpha E_{xx}, \; I_{xx} = D_{xx} - \alpha F_{xx}, \; J_{xx} = F_{xx} - \alpha H_{xx} \tag{45}$$

By substituting for the second derivative of Q from Eq. (28) into Eq. (33), and using Eqs. (33) and (34) the following expression for the nonlocal shear force is derived

$$
\begin{aligned}
Q =& \bar{A}_{xx}(\frac{\partial w}{\partial x}+\psi)-(E_{15}-\beta F_{15})\frac{\partial \phi}{\partial x}+\mu(E\frac{\partial^3 w}{\partial x^3}-\alpha\frac{\partial^3 P}{\partial x^3}-\frac{\partial q}{\partial x}) \\
&+\mu(I_0\frac{\partial^3 w}{\partial x\partial t^2}+\alpha I_3\frac{\partial^4 u}{\partial x^2\partial t^2}+\alpha I_4\frac{\partial^4 u}{\partial x^2\partial t^2}-\alpha I_6(\frac{\partial^5 w}{\partial x^3\partial t^2}+\frac{\partial^4 w}{\partial x^2\partial t^2}))
\end{aligned} \tag{46}
$$

where

$$
\bar{A}_{xz}=A^*_{xz}-\beta I^*_{xz},\ A^*_{xz}=A_{xz}-\beta D_{xz},\ I^*_{xz}=D_{xz}-\beta F_{xz} \tag{47}
$$

Now we use M and Q from Eqs. (44) and (46) and the identity

$$
\alpha\frac{\partial^2}{\partial x^2}(P-\mu\frac{\partial^2 P}{\partial x^2})=\alpha(E_{xx}\frac{\partial^3 u}{\partial x^3}+F_{xx}\frac{\partial^3\psi}{\partial x^3}-\alpha H_{xx}(\frac{\partial^3\psi}{\partial x^3}+\frac{\partial^4 w}{\partial x^4})+F_{31}\frac{\partial^2\phi}{\partial x^2}) \tag{48}
$$

It must be cited that inserting Eq. (29) into Eqs. (35) and (36), does not provide an explicit expressions for D_x and D_z. To overcome this problem, by using Eqs. (35) and (36), Eq. (29) can be re-expressed in terms of u, w, ψ and ϕ. Finally, based on third-order beam theory, the nonlocal equations of motion for a FG piezoelectric nanobeam can be obtained by substituting for N, M and Q from Eqs. (42), (44) and (46) into Eqs. (26)-(28) as follows

$$
\begin{aligned}
&A_{xx}\frac{\partial^2 u}{\partial x^2}+K_{xx}\frac{\partial^2\psi}{\partial x^2}-\alpha E_{xx}\frac{\partial^3 w}{\partial x^3}+A^e_{31}\frac{\partial\phi}{\partial x}+\mu(-\frac{\partial^2 f}{\partial x^2}+I_0\frac{\partial^4 u}{\partial x^2\partial t^2}+I_1\frac{\partial^4\psi}{\partial x^2\partial t^2} \\
&-\alpha I_3\frac{\partial^4\psi}{\partial x^2\partial t^2}-\alpha I_3\frac{\partial^5 w}{\partial x^3\partial t^2})+f-I_0\frac{\partial^2 u}{\partial t^2}-\hat{I}_1\frac{\partial^2\psi}{\partial t^2}+\alpha I_3\frac{\partial^3 w}{\partial x\partial t^2}=0
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
&K_{xx}\frac{\partial^2 u}{\partial x^2}+I_{xx}\frac{\partial^2\psi}{\partial x^2}-\alpha J_{xx}(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^3 w}{\partial x^3})-\bar{A}_{xz}\left(\varphi+\frac{\partial w}{\partial x}\right)+(E_{31}-\alpha F_{31})\phi-\hat{I}_2\frac{\partial^2\psi}{\partial t^2} \\
&+\alpha\hat{I}_4(\frac{\partial^2\psi}{\partial t^2}+\frac{\partial^3 w}{\partial x\partial t^2})-\hat{I}_1\frac{\partial^2 u}{\partial t^2}+(E_{15}-\beta F_{15})\frac{\partial\phi}{\partial x}+\mu(\hat{I}_1\frac{\partial^4 u}{\partial x^2\partial t^2}+\hat{I}_2\frac{\partial^4\psi}{\partial x^2\partial t^2} \\
&-\alpha\hat{I}_4(\frac{\partial^4\psi}{\partial x^2\partial t^2}+\frac{\partial^5 w}{\partial x^3\partial t^2}))=0
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
&\bar{A}_{xz}\left(\frac{\partial\psi}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+\mu(N_E\frac{\partial^4 w}{\partial x^4}-\frac{\partial^2 q}{\partial x^2})+q-(N_E)\frac{\partial^2 w}{\partial x^2}-(E_{15}-\beta F_{15})\frac{\partial\phi}{\partial x}+\alpha(E_{xx}\frac{\partial^3 u}{\partial x^3} \\
&+J_{xx}\frac{\partial^3\psi}{\partial x^3}-\alpha H_{xx}\frac{\partial^4 w}{\partial x^4}+F_{31}\frac{\partial^2\phi}{\partial x^2})-I_0\frac{\partial^2 w}{\partial t^2}-\alpha I_3\frac{\partial^3 u}{\partial x\partial t^2}-\alpha I_4\frac{\partial^3\psi}{\partial t^2\partial x}+\alpha^2 I_6(\frac{\partial^3\psi}{\partial t^2\partial x}+\frac{\partial^4 w}{\partial t^2\partial x^2}) \\
&+\mu(I_0\frac{\partial^4 w}{\partial x^2\partial t^2}+\alpha I_3\frac{\partial^5 u}{\partial x^3\partial t^2}+\alpha I_4\frac{\partial^5\psi}{\partial t^2\partial x^3}-\alpha^2 I_6(\frac{\partial^5\psi}{\partial t^2\partial x^3}+\frac{\partial^6 w}{\partial t^2\partial x^4}))=0
\end{aligned} \tag{51}
$$

$$
(E_{15}-\beta F_{15})(\frac{\partial^2 w}{\partial x^2}+\frac{\partial\psi}{\partial x})+F_{11}\frac{\partial^2\phi}{\partial x^2}+A^e_{31}\frac{\partial u}{\partial x}+(E_{31}-\alpha F_{31})\frac{\partial\psi}{\partial x}-\alpha F_{31}\frac{\partial^2 w}{\partial x^2}-F_{33}\phi=0 \tag{52}
$$

3. Solution procedure

Here, on the basis the Navier method, an analytical solution of the governing equations for free vibration of a simply supported FGP nanobeam is presented. To satisfy governing equations of motion and the simply supported boundary condition, the displacement variables are adopted to be of the form

$$u(x,t)=\sum_{n=1}^{\infty}U_n\cos(\frac{n\pi}{L}x)e^{i\omega_n t} \tag{53}$$

$$w(x,t)=\sum_{n=1}^{\infty}W_n\sin(\frac{n\pi}{L}x)e^{i\omega_n t} \tag{54}$$

$$\psi(x,t)=\sum_{n=1}^{\infty}\Psi_n\cos(\frac{n\pi}{L}x)e^{i\omega_n t} \tag{55}$$

$$\phi(x,t)=\sum_{n=1}^{\infty}\Phi_n\sin(\frac{n\pi}{L}x)e^{i\omega_n t} \tag{56}$$

where U_n, W_n, Ψ_n and Φ_n are the unknown Fourier coefficients to be determined for each n value. The boundary conditions for simply supported FGP beam can be identified as

$$\begin{gathered} u(0)=0, \qquad \frac{\partial^2 w}{\partial x^2}(L)=0; \quad W(0)=w(L)=0 \\ \frac{\partial\psi}{\partial x}(0)=\frac{\partial\psi}{\partial x}(L)=0, \quad \phi(0)=\phi(L)=0 \end{gathered} \tag{57}$$

Substituting Eqs. (53)-(56) into Eqs. (49)-(52) respectively, leads to

$$\begin{aligned} &(-A_{xx}(\frac{n\pi}{l})^2+I_0(1+\mu(\frac{n\pi}{l})^2)\omega^2{}_n)U_n+(-K_{xx}(\frac{n\pi}{l})^2+\hat{I}_1((1+\mu(\frac{n\pi}{l})^2)\omega^2{}_n))\psi_n \\ &+(\alpha E_{xx}(\frac{n\pi}{l})^3-\alpha I_3(\frac{n\pi}{l})\omega^2{}_n-\alpha I_3\mu(\frac{n\pi}{l})^3\omega^2{}_n)W_n+(-A^e_{31}(\frac{n\pi}{l}))\phi_n=0 \end{aligned} \tag{58}$$

$$\begin{aligned} &(-K_{xx}(\frac{n\pi}{l})^2+\hat{I}_1\omega^2{}_n+\mu\hat{I}_1\omega^2{}_n(\frac{n\pi}{l})^2)U_n+(-I_{xx}(\frac{n\pi}{l})^2+\alpha J_{xx}(\frac{n\pi}{l})^2-\bar{A}_{xz}+\hat{I}_2\omega^2{}_n-\alpha\hat{I}_4\omega^2{}_n \\ &+\mu((\hat{I}_2-\alpha\hat{I}_4)\omega^2{}_n(\frac{n\pi}{l})^2)\psi_n+(\alpha J_{xx}(\frac{n\pi}{l})^3-\bar{A}_{xz}(\frac{n\pi}{l})-\alpha\hat{I}_4\omega^2{}_n(\frac{n\pi}{l})+\mu\alpha\hat{I}_4\omega^2{}_n(\frac{n\pi}{l})^3)W_n \\ &+((E_{31}-\alpha F_{31})+(E_{15}-\beta F_{15})(\frac{n\pi}{l}))\phi_n=0 \end{aligned} \tag{59}$$

$$(\alpha E_{xx}(\frac{n\pi}{l})^3-\alpha I_3(\frac{n\pi}{l})\omega^2{}_n-\mu\alpha I_3(\frac{n\pi}{l})^3\omega^2{}_n)U_n+(-\bar{A}_{xz}(\frac{n\pi}{l})+J_{xx}(\frac{n\pi}{l})^3-\alpha\hat{I}_4(\frac{n\pi}{l})\omega^2{}_n \tag{60}$$

$$
\begin{aligned}
&+\mu(-\alpha\hat{I}_4(\frac{n\pi}{l})^3\omega^2{}_n))\psi_n+(N_E(\frac{n\pi}{l})^2(1+\mu(\frac{n\pi}{l})^2)-\bar{A}_{xz}(\frac{n\pi}{l})^2-\alpha^2(\frac{n\pi}{l})^4+I_0(\frac{n\pi}{l})^2 \\
&+\alpha^2 I_6(\frac{n\pi}{l})^2\omega^2{}_n+\mu(I_0(\frac{n\pi}{l})^2\omega^2{}_n+\alpha^2 I_6(\frac{n\pi}{l})^4\omega^2{}_n))W_n+(-(E_{15}-\beta F_{15})(\frac{n\pi}{l})-F_{31}(\frac{n\pi}{l})^2)\phi_n=0
\end{aligned} \tag{60}
$$

$$
\begin{aligned}
&\left(-A_{31}^e(\frac{n\pi}{L})\right)U_n-\left(((E_{15}-\beta F_{15})-\alpha F_{31})(\frac{n\pi}{L})^2\right)W_n-\left((E_{15}-\beta F_{15})+(E_{31}-\alpha F_{31})(\frac{n\pi}{L})\right)\Psi_n \\
&-\left(F_{11}(\frac{n\pi}{L})^2+F_{33}\right)\Phi_n=0
\end{aligned} \tag{61}
$$

By setting the determinant of the coefficient matrix of the above equations, the nontrivial analytical solutions can be obtained from the following equations

$$
\left\{[K]-\bar{\omega}^2[M]\right\}\begin{Bmatrix} U_n \\ W_n \\ \Psi_n \\ \Phi_n \end{Bmatrix}=0 \tag{62}
$$

where $[K]$ denotes the stiffness matrix, and $[M]$ is the mass matrix. By setting this polynomial to zero, we can find natural frequencies $\bar{\omega}_n$ of the FGP nanobeam exposed to electrical loading.

4. Results and discussion

In this section, several numerical examples are provided for the electro-mechanical free vibration characteristics of FGPM nanobeams. To achieve this goal, the nonlocal FGP beam made of PZT-4 and PZT-5H, with electro-mechanical material properties listed in Table 1, is supposed.

The beam geometry has the following dimensions: L (length) = 10 nm and h (thickness) = varied. Also, the following relation is described to calculate the non-dimensional natural frequencies

$$
\bar{\omega}=\omega L^2\sqrt{(\frac{\rho A}{c_{11}I})_{\text{PZT-4}}} \tag{63}
$$

in which $I = h^3/12$ is the moment of inertia of the cross section of the beam. For verification purpose the frequency results are compared with those of nonlocal FGM Timoshenko beams presented by Rahmani and Pedram (2014), due to the fact that any numerical results for the free vibration of FGP nanobeams based on the nonlocal elasticity theory are not existing yet. In this work, the material selection is performed as follows: E_m = 70 GPa, v_m = 0.3, ρ_m = 7800 kg m^{-3} for Steel and E_c = 390 GPa, v_c = 0.24, ρ_c = 3960 kg m^{-3} for Alumina. Therefore, Table 2 presents the fundamental frequency of S-S FG nanobeams in comparison to those of Rahmani and Pedram (2014).

The influences of several parameters including external electric voltage (V), material

Table 1 Electro-mechanical coefficients of material properties for PZT-4 and PZT-5H (Doroushi *et al*. 2011)

Properties	PZT-4	PZT-5H
c_{11} (GPa)	81.3	60.6
c_{55} (GPa)	25.6	23.0
e_{31} (cm^{-2})	-10.0	-16.604
e15 (cm^{-2})	40.3248	44.9046
k_{11} (C^{2}m^{-2}N^{-1})	0.6712e-8	1.5027e-8
k_{33} (C^{2}m^{-2}N^{-1})	1.0275e-8	2.554e-8
ρ (kgm^{-3})	7500	7500

Table 2 Comparison of the non-dimensional fundamental frequency for a S-S FG nanobeam with various power-law index (L/h = 20)

μ (nm^2)	p = 0		p = 0.5		p = 1		p = 5	
	TBT (Rahmani and Pedram 2014)	Present RBT	TBT (Rahmani and Pedram 2014)	Present RBT	TBT (Rahmani and Pedram 2014)	Present RBT	TBT (Rahmani and Pedram 2014)	Present RBT
0	9.8296	9.829570	7.7149	7.71546	6.9676	6.967613	5.9172	5.916152
1	9.3777	9.377686	7.3602	7.36078	6.6473	6.647300	5.6452	5.644175
2	8.9829	8.982894	7.0504	7.05090	6.3674	6.367454	5.4075	5.406561
3	8.6341	8.634103	6.7766	6.77714	6.1202	6.120217	5.1975	5.196632
4	8.3230	8.323021	6.5325	6.53296	5.8997	5.899708	5.0103	5.009400

Table 3 Influence of external electric voltage and material composition on the 1st non-dimensional frequency of a S-S FGP nanobeam (L/h = 20)

μ		Gradient index						
		p = 0	p = 0.2	p = 0.5	p = 1	p = 2	p = 5	p = 10
0	V = -0.5	10.9593	10.7792	10.6445	10.5545	10.5024	10.4459	10.3829
	V = -0.25	10.6907	10.4755	10.3058	10.1815	10.0958	10.0051	9.92464
	V = 0	10.4152	10.1627	9.95565	9.79425	9.67212	9.54393	9.44416
	V = +0.25	10.1322	9.83995	9.59269	9.39106	9.22901	9.05931	8.93788
	V = +0.5	9.84104	9.50628	9.21545	8.96976	8.76352	8.54726	8.40115
1	V = -0.5	10.5054	10.3399	10.2177	10.1381	10.0944	10.0467	9.98964
	V = -0.25	10.2248	10.0229	9.86442	9.74912	9.67066	9.58748	9.51244
	V = 0	9.93640	9.69548	9.49798	9.34399	9.22748	9.10518	9.0100
	V = +0.25	9.63934	9.35666	9.11681	8.92047	8.76190	8.59586	8.47783
	V = +0.5	9.33283	9.00510	8.71900	8.47582	8.27016	8.05440	7.90994
2	V = -0.5	10.1106	9.95814	9.84715	9.77671	9.74058	9.70061	9.64891
	V = -0.25	9.81882	9.62857	9.48004	9.37277	9.30072	9.22422	9.15397

Table 3 Continued

μ		Gradient index						
		p = 0	p = 0.2	p = 0.5	p = 1	p = 2	p = 5	p = 10
2	V = 0	9.51809	9.28731	9.09812	8.95061	8.83901	8.72186	8.63068
	V = +0.25	9.20754	8.93302	8.69945	8.50754	8.35181	8.18874	8.07355
	V = +0.5	8.88615	8.56408	8.28162	8.04008	7.83437	7.61839	7.47501
3	V = -0.5	9.76349	9.62270	9.52172	9.45958	9.43024	9.39729	9.35036
	V = -0.25	9.46100	9.28122	9.14154	9.04147	8.97519	8.90469	8.83871
	V = 0	9.14852	8.92669	8.74485	8.60308	8.49580	8.38320	8.29557
	V = +0.25	8.82497	8.55749	8.32930	8.14111	7.98770	7.82705	7.71427
	V = +0.5	8.48911	8.17162	7.89189	7.65129	7.44500	7.22822	7.08545

composition and nonlocal parameter (μ) on the first three non-dimensional frequencies of the simply supported higher order FG nanobeams at L/h = 20 are presented in Tables 3-5. It is observable that with the increase of nonlocal parameter the natural frequencies of FG nanobeam reduces for all external voltages due to the fact that existence of nonlocality weakens the beam. In addition, it is found that as the gradient index arose the non-dimensional frequencies of piezoelectric FG nanobeam decrease, especially for smaller gradient indexes. Also, it is concluded that negative values of external voltage produces higher frequencies compared to those of positive voltages.

The variations of the 1st fundamental frequency of FGP nanobeams versus the gradient index for different external voltages and nonlocal parameters at L/h = 20 are depicted in Fig. 2. It is observed from the figure that the dimensionless natural frequency reduces vigorously for lower values of gradient index, and then reduces monotonically for higher values of gradient index. Also, the frequency variations for positive voltages is more sensible than that of negative one.

Table 4 Influence of external electric voltage and material composition on the 2nd non-dimensional frequency of a S-S FGP nanobeam (L/h = 20)

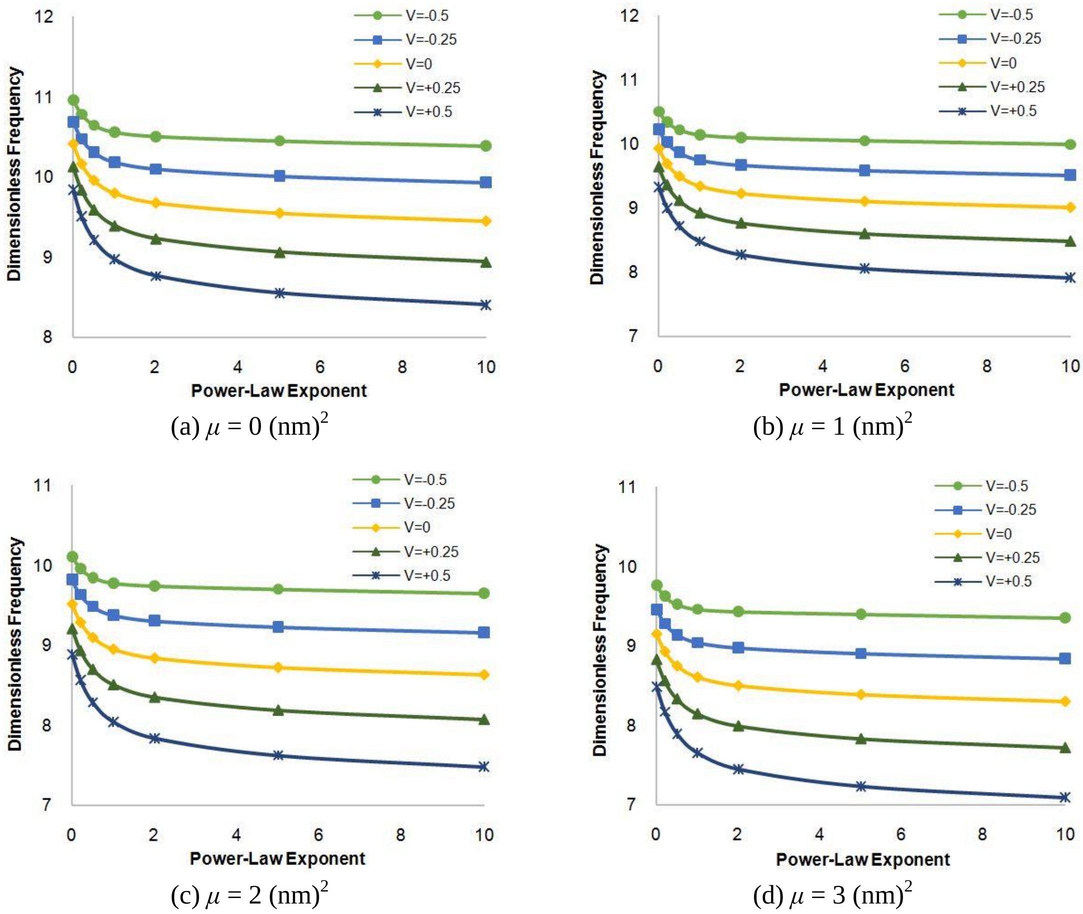
μ		Gradient index						
		p = 0	p = 0.2	p = 0.5	p = 1	p = 2	p = 5	p = 10
0	V = -0.5	41.9836	41.0434	40.2935	39.7276	39.3173	38.8856	38.5307
	V = -0.25	41.7073	40.7295	39.9418	39.3386	38.8915	38.4219	38.0474
	V = 0	41.4292	40.4132	39.5871	38.9457	38.4610	37.9525	37.5580
	V = +0.25	41.1492	40.0943	39.2291	38.5488	38.0256	37.4773	37.0620
	V = +0.5	40.8673	39.7729	38.8678	38.1477	37.5851	36.9959	36.5594
1	V = -0.5	35.7326	34.9612	34.3511	33.8965	33.5732	33.2325	32.9447
	V = -0.25	35.4075	34.5922	33.9379	33.4397	33.0735	32.6887	32.3782
	V = 0	35.0795	34.2192	33.5197	32.9766	32.5662	32.1357	31.8016
	V = +0.25	34.7483	33.8420	33.0962	32.5069	32.0508	31.573	31.2143
	V = +0.5	34.4140	33.4606	32.6671	32.0303	31.5270	31.0001	30.6158

Table 4 Continued

μ		Gradient index						
		p = 0	p = 0.2	p = 0.5	p = 1	p = 2	p = 5	p = 10
2	V = -0.5	31.7072	31.0479	30.5309	30.1509	29.8864	29.6071	29.3640
	V = -0.25	31.3405	30.6317	30.0653	29.6364	29.3240	28.9954	28.7270
	V = 0	30.9694	30.2099	29.5923	29.1129	28.7505	28.3705	28.0755
	V = +0.25	30.5938	29.7820	29.1117	28.5797	28.1654	27.7315	27.4086
	V = +0.5	30.2135	29.3479	28.6231	28.0365	27.5679	27.0774	26.7250
3	V = -0.5	28.8445	28.2671	27.8184	27.4935	27.2728	27.0391	26.8288
	V = -0.25	28.4409	27.8094	27.3066	26.9283	26.6553	26.3678	26.1301
	V = 0	28.0314	27.3440	26.7850	26.3510	26.0231	25.6791	25.4121
	V = +0.25	27.6159	26.8705	26.2531	25.7608	25.3752	24.9713	24.6733
	V = +0.5	27.1940	26.3885	25.7101	25.1567	24.7102	24.2429	23.9116

Table 5 Influence of external electric voltage and material composition on the 3rd non-dimensional frequency of a S-S FGP nanobeam (L/h = 20)

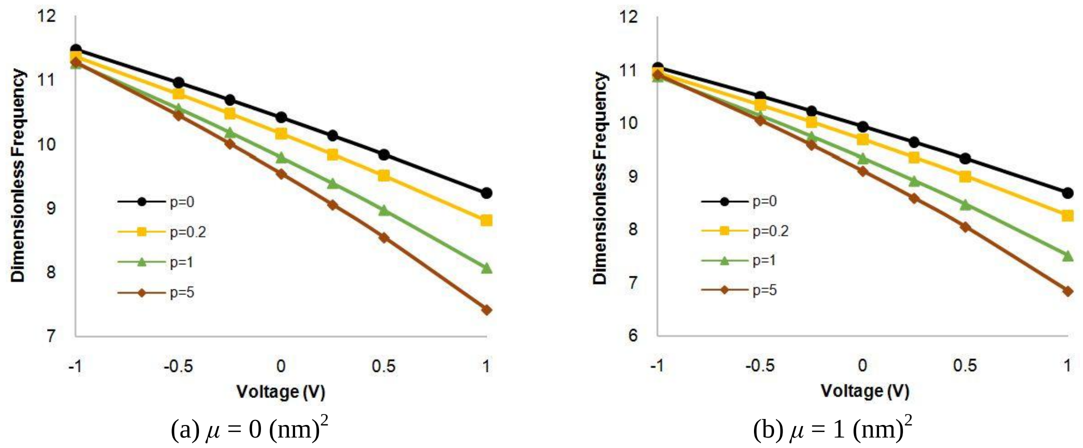
μ		Gradient index						
		p = 0	p = 0.2	p = 0.5	p = 1	p = 2	p = 5	p = 10
0	V = -0.5	92.9812	90.7489	88.9701	87.6146	86.6103	85.5572	84.7236
	V = -0.25	92.7036	90.4331	88.6159	87.2225	86.1807	85.0889	84.2353
	V = 0	92.4252	90.1162	88.2603	86.8285	85.7489	84.6181	83.7443
	V = +0.25	92.1460	89.7981	87.9033	86.4328	85.3150	84.1446	83.2503
	V = +0.5	91.8659	89.4790	87.5449	86.0353	84.8788	83.6684	82.7534
1	V = -0.5	68.0224	66.4467	65.2013	64.2634	63.5803	62.8631	62.2818
	V = -0.25	67.6424	66.0148	64.7172	63.7277	62.9938	62.2243	61.6160
	V = 0	67.2604	65.5800	64.2295	63.1875	62.4018	61.5789	60.9430
	V = +0.25	66.8761	65.1423	63.7380	62.6426	61.8042	60.9266	60.2624
	V = +0.5	66.4896	64.7016	63.2427	62.0930	61.2007	60.2672	59.5741
2	V = -0.5	56.3892	55.1297	54.1424	53.4084	52.8838	52.3321	51.8732
	V = -0.25	55.9303	54.6083	53.5585	52.7626	52.1772	51.5630	51.0719
	V = 0	55.4676	54.0819	52.9681	52.1088	51.4609	50.7823	50.2579
	V = +0.25	55.0010	53.5502	52.3711	51.4468	50.7345	49.9893	49.4304
	V = +0.5	54.5305	53.0133	51.7672	50.7761	49.9976	49.1835	48.5889
3	V = -0.5	49.3359	48.2739	47.4487	46.8436	46.4201	45.9739	45.5919
	V = -0.25	48.8107	47.6775	46.7813	46.1059	45.6135	45.0965	44.6781
	V = 0	48.2798	47.0737	46.1042	45.3563	44.7924	44.2016	43.7452
	V = +0.25	47.7431	46.4619	45.4171	44.5941	43.9559	43.2883	42.7920
	V = +0.5	47.2002	45.8420	44.7193	43.8186	43.1032	42.3553	41.8171



(a) $\mu = 0$ (nm)^2 (b) $\mu = 1$ (nm)^2

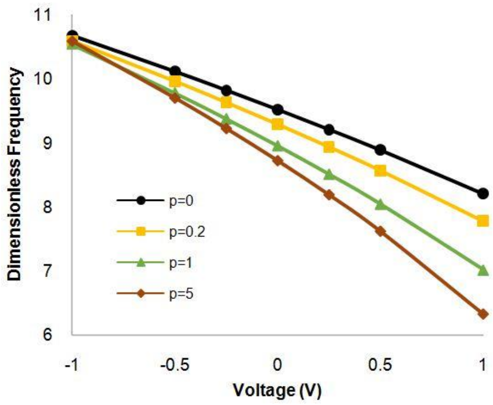
(c) $\mu = 2$ (nm)^2 (d) $\mu = 3$ (nm)^2

Fig. 2 Effect of external electric voltage on the dimensionless frequency of the S-S FGP nanobeam with respect to gradient index for different values of nonlocal parameters (L/h = 20)

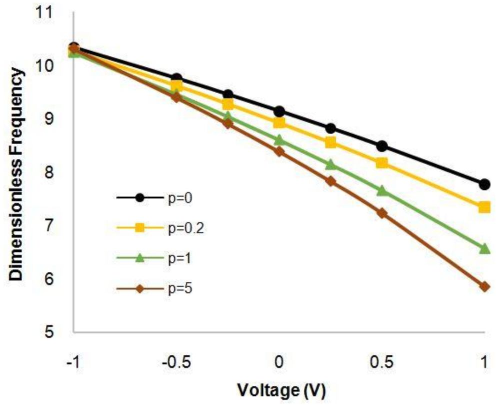


(a) $\mu = 0$ (nm)^2 (b) $\mu = 1$ (nm)^2

Fig. 3 The variation of dimensionless frequency of the S-S FGP nanobeam with respect to external voltage for different values of nonlocal parameters and gradient indexes (L/h = 20)

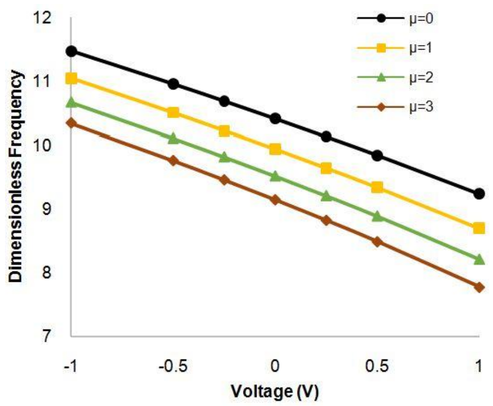


(c) μ = 2 (nm)^2

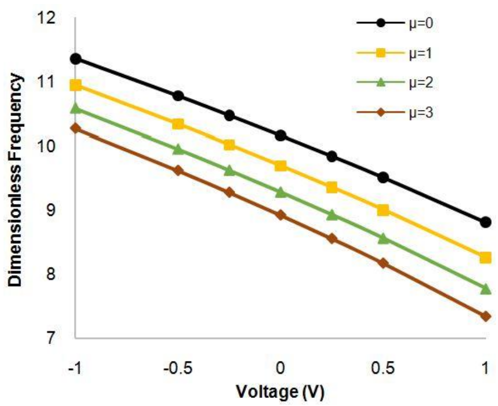


(d) μ = 3 (nm)^2

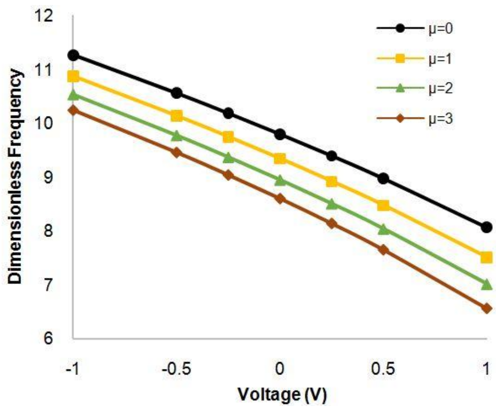
Fig. 3 Continued



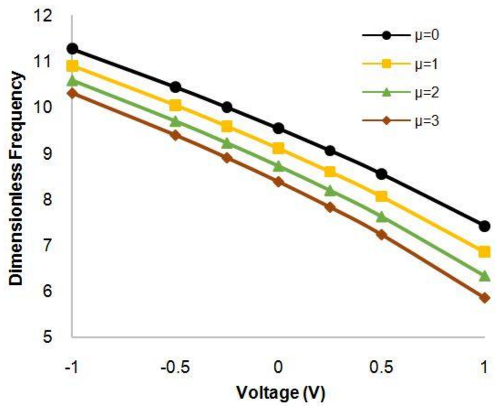
(a) p = 0



(b) p = 0.2



(c) p = 1



(d) p = 5

Fig. 4 The variation of dimensionless frequency of the S-S FGP nanobeam with respect to external voltage for different values of nonlocal parameters and gradient indexes (L/h = 20)

The variations of the first non-dimensional frequency of the simply supported FG nanobeams versus external voltage for various values of nonlocal parameter and gradient index are plotted in Fig. 3. It is found that external voltage shows a decreasing effect on the natural frequencies of FG

nanobeams when it changes from negative values to positive one. Therefore, when the voltage value increases from negative to positive the natural frequency reduces, but difference between the curves rises. So, the frequency results for negative voltages are more close to each other.

Fig. 4 shows the effect of nonlocal parameter on the variations of the dimensionless frequency of nonlocal FG beams with respect to external voltage at L/h = 20 for various gradient indexes. An important observation is that, the nonlocal parameter effect is not dependent on the external voltage values, since the difference between local and nonlocal frequency curves stays constant. Therefore, for all values of nonlocal parameter, with the increase of external voltage from negative to positive values the natural frequencies reduce with a same manner.

Fig. 5 demonstrate the variations of the non-dimensional frequency of piezoelectric FG nanobeam with respect to slenderness ratio for gradient index p = 0.2 and nonlocal parameter μ = 2. The most important observation from the figure is that, external voltage shows an increasing influence on natural frequencies of FGP nonlocal beams for negative values of external voltage and a decreasing effect for positive voltage values. Also, it is found that the variations of dimensionless frequency is approximately independent of slenderness ratio when the external voltages is set to zero V = 0. Also, it is seen that curvature of the lines for higher values of external voltage (both negative and positive) is more than lower voltages which means FG nanobeam is more affected by the larger voltages.

The influence of mode number on the non-dimensional frequency of nonlocal piezoelectric FGM beams at p = 1 and L/h = 20 is presented in Fig. 6. It is deduced from this figure that the effect of nonlocality on the lower mode numbers of FGP nanobeams is less than higher modes. So the difference of obtained frequencies between the various values of nonlocal parameters rises with the increase of mode number. Also, it is found that effect of electric voltage on lower modes is more than higher modes.

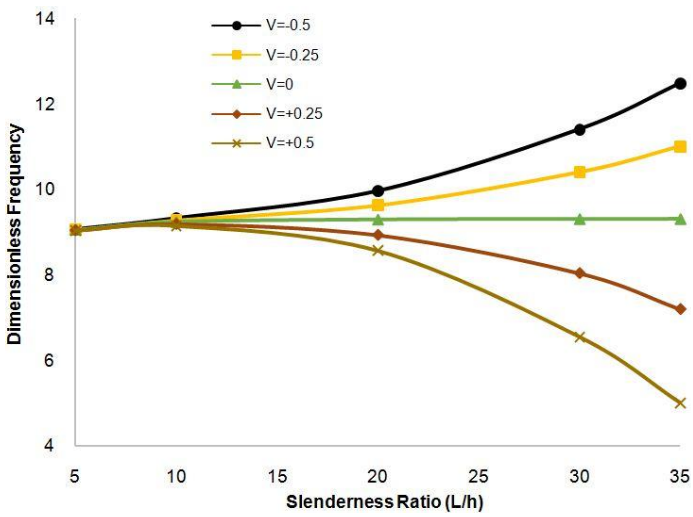
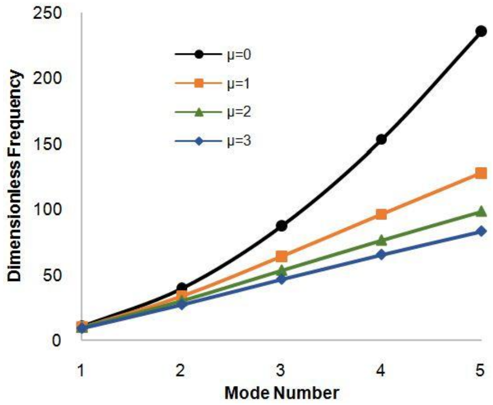
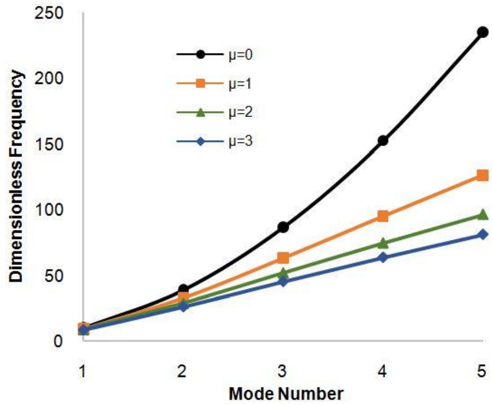


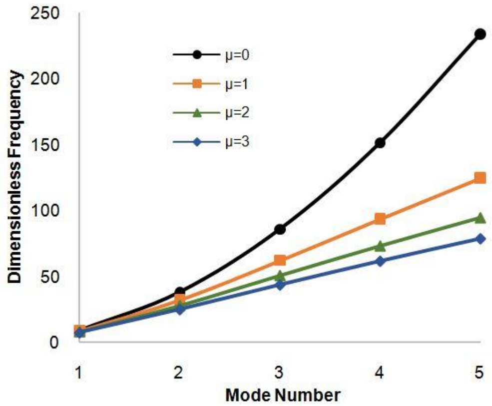
Fig. 5 The variation of dimensionless frequency of the S-S FGP nanobeam with respect to slenderness ratio for different values of external voltage (μ = 2, p = 0.2)



(a) V = -0.5



(b) V = 0



(c) V = +0.5

Fig. 6 Effect of mode number on the dimensionless frequency of the S-S FGP nanobeam for different values of external voltages (p = 1, L/h = 20)

5. Conclusions

The present study develops a nonlocal higher order beam model for free vibration analysis of piezoelectric FG nanobeams. Eringen's nonlocal elasticity theory is adopted to capture the small size effects and the nonolocal governing equations are solved using Navier solution method. Eelectro-mechanical properties of the FGP nanobeams are supposed to be position dependent based on power-law model. Correctness of the results is checked with available data in the literature. Several numerical examples indicate the influences of some parameters including external electric voltage, gradient index, nonlocal parameter, slenderness ratio and mode number on the natural frequencies of nonlocal FGM beams. It is seen that presence of nonlocality leads to reduction in both rigidity of the beam and natural frequencies. Also, depending on the sign of the voltage the external electric voltage shows both decreasing and increasing effects on the natural frequencies. In addition, it is concluded that, the influence of nonlocality is independent of electric voltage value.

References

Ansari, R., Pourashraf, T. and Gholami, R. (2015), "An exact solution for the nonlinear forced vibration of functionally graded nanobeams in thermal environment based on surface elasticity theory", *Thin-Wall. Struct.*, **93**, 169-176.

Aydogdu, M. (2009), "A general nonlocal beam theory: its application to nanobeam bending, buckling and vibration", *Physica E: Low-dimen. Syst. Nanostruct.*, **41**(9), 1651-1655.

Dehrouyeh-Semnani, A.M. and Nikkhah-Bahrami, M. (2015), "The influence of size-dependent shear deformation on mechanical behavior of microstructures-dependent beam based on modified couple stress theory", *Compos. Struct.*, **123**, 325-336.

Doroushi, A., Eslami, M.R. and Komeili, A. (2011), "Vibration analysis and transient response of an FGPM beam under thermo-electro-mechanical loads using higher-order shear deformation theory", *J. Intel. Mater. Syst. Struct.*, **22**(3), 231-243.

Ebrahimi, F. and Salari, E. (2015a), "A semi-analytical method for vibrational and buckling analysis of functionally graded nanobeams considering the physical neutral axis position", *CMES: Comput. Model. Eng. Sci.*, **105**(2), 151-181

Ebrahimi, F. and Salari, E. (2015b), "Size-dependent free flexural vibrational behavior of functionally graded nanobeams using semi-analytical differential transform method", *Compos. Part B: Eng.*, **79**, 156-169.

Ebrahimi, F., Ghadiri, M., Salari, E., Hoseini, S.A.H. and Shaghaghi, G.R. (2015), "Application of the differential transformation method for nonlocal vibration analysis of functionally graded nanobeams", *J. Mech. Sci. Technol.*, **29**(3), 1207-1215.

Eltaher, M.A., Emam, S.A. and Mahmoud, F.F. (2012), "Free vibration analysis of functionally graded size-dependent nanobeams", *Appl. Math. Comput.*, **218**(14), 7406-7420.

Eltaher, M.A., Alshorbagy, A.E. and Mahmoud, F.F. (2013a), "Determination of neutral axis position and its effect on natural frequencies of functionally graded macro/nanobeams", *Composite Structures*, **99**, 193-201.

Eltaher, M.A., Emam, S.A. and Mahmoud, F.F. (2013b), "Static and stability analysis of nonlocal functionally graded nanobeams", *Composite Structures*, **96**, 82-88.

Eringen, A.C. (1972), "Nonlocal polar elastic continua", *Int. J. Eng. Sci.*, **10**(1), 1-16.

Eringen, A.C. (1983), "On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves", *J. Appl. Phys.*, **54**(9), 4703-4710.

Eringen, A.C. and Edelen, D.G.B. (1972), "On nonlocal elasticity", *Int. J. Eng. Sci.*, **10**(3), 233-248.

Filippi, M., Carrera, E. and Zenkour, A.M. (2015), "Static analyses of FGM beams by various theories and finite elements", *Compos. Part B: Eng.*, **72**, 1-9.

Hosseini-Hashemi, S., Nahas, I., Fakher, M. and Nazemnezhad, R. (2014), "Surface effects on free vibration of piezoelectric functionally graded nanobeams using nonlocal elasticity", *Acta Mechanica*, **225**(6), 1555-1564.

Kiani, Y., Rezaei, M., Taheri, S. and Eslami, M.R. (2011), "Thermo-electrical buckling of piezoelectric functionally graded material Timoshenko beams", *Int. J. Mech. Mater. Des.*, **7**(3), 185-197.

Komijani, M., Kiani, Y., Esfahani, S.E. and Eslami, M.R. (2013), "Vibration of thermo-electrically post-buckled rectangular functionally graded piezoelectric beams", *Compos. Struct.*, **98**, 143-152.

Lezgy-Nazargah, M. (2015), "A three-dimensional exact state-space solution for cylindrical bending of continuously non-homogenous piezoelectric laminated plates with arbitrary gradient composition", *Arch. Mech.*, **67**(1), 25-51.

Lezgy-Nazargah, M. (2016), "A three-dimensional Peano series solution for the vibration of functionally graded piezoelectric laminates in cylindrical bending", *Scientia Iranica A*, **23**(3), 788-801.

Lezgy-Nazargah, M., Vidal, P. and Polit, O. (2013), "An efficient finite element model for static and dynamic analyses of functionally graded piezoelectric beams", *Compos. Struct.*, **104**, 71-84.

Li, L. and Hu, Y. (2017a), "Torsional vibration of bi-directional functionally graded nanotubes based on

nonlocal elasticity theory", *Compos*. *Struct*., **172**, 242-250.
Li, L. and Hu, Y. (2017b), "Post-buckling analysis of functionally graded nanobeams incorporating nonlocal stress and microstructure-dependent strain gradient effects", *Int*. *J*. *Mech*. *Sci*., **120**, 159-170.
Li, Y.S., Feng, W.J. and Cai, Z.Y. (2014), "Bending and free vibration of functionally graded piezoelectric beam based on modified strain gradient theory", *Compos*. *Struct*., **115**, 41-50.
Nazemnezhad, R. and Hosseini-Hashemi, S. (2014), "Nonlocal nonlinear free vibration of functionally graded nanobeams", *Compos*. *Struct*., **110**, 192-199.
Rahmani, O. and Jandaghian, A.A. (2015), "Buckling analysis of functionally graded nanobeams based on a nonlocal third-order shear deformation theory", *Appl*. *Phys*. *A*, **119**(3), 1019-1032.
Rahmani, O. and Pedram, O. (2014), "Analysis and modeling the size effect on vibration of functionally graded nanobeams based on nonlocal Timoshenko beam theory", *Int*. *J*. *Eng*. *Sci*., **77**, 55-70.
Reddy, J.N. (2007), "Nonlocal theories for bending, buckling and vibration of beams", *Int*. *J*. *Eng*. *Sci*., **45**(2), 288-307.
Sharabiani, P.A. and Yazdi, M.R.H. (2013), "Nonlinear free vibrations of functionally graded nanobeams with surface effects", *Compos*. *Part B*: *Eng*., **45**(1), 581-586.
Shegokar, N.L. and Lal, A. (2014), "Stochastic finite element nonlinear free vibration analysis of piezoelectric functionally graded materials beam subjected to thermo-piezoelectric loadings with material uncertainties", *Meccanica*, **49**(5), 1039-1068.
Shi, Z.F. and Chen, Y. (2004), "Functionally graded piezoelectric cantilever beam under load", *Arch*. *Appl*. *Mech*., **74**(3-4), 237-247.
Şimşek, M. and Yurtcu, H.H. (2013), "Analytical solutions for bending and buckling of functionally graded nanobeams based on the nonlocal Timoshenko beam theory", *Compos*. *Struct*., **97**, 378-386.
Soldatos, K.P. (1992), "A transverse shear deformation theory for homogeneous monoclinic plates", *Acta Mechanica*, **94**(3-4), 195-220.
Su, Z., Jin, G. and Ye, T. (2016), "Vibration analysis and transient response of a functionally graded piezoelectric curved beam with general boundary conditions", *Smart Mater*. *Struct*., **25**(6), 065003.
Touratier, M. (1991), "An efficient standard plate theory", *Int*. *J*. *Eng*. *Sci*., **29**(8), 901-916.
Uymaz, B. (2013), "Forced vibration analysis of functionally graded beams using nonlocal elasticity", *Compos*. *Struct*., **105**, 227-239.
Yang, J. and Xiang, H.J. (2007), "Thermo-electro-mechanical characteristics of functionally graded piezoelectric actuators", *Smart Mater*. *Struct*., **16**(3), 784.
Zhu, X. and Li, L. (2017), "Twisting statics of functionally graded nanotubes using Eringen's nonlocal integral model", *Compos*. *Struct*., **178**, 87-96.

JL