Advances in Nano Research, *Vol. 5, No. 2 (2017) 179-192*
DOI: https://doi.org/10.12989/anr.2017.5.2.179

Free vibration analysis of double walled carbon nanotubes embedded in an elastic medium with initial imperfection

Javad Ehyaei* and Mohsen Daman

Department of Mechanical Engineering, Faculty of Engineering, Imam Khomeini International University, Qazvin, Iran

(*Received October 08, 2016, Revised December 13, 2016, Accepted April 12, 2017*)

Abstract. The transverse vibration of double walled carbon nanotube (DWCNT) embedded in elastic medium with an initial imperfection is considered. In this paper, Timoshenko beam theory is employed. However the nonlocal theory is used for modeling the nano scale of nanotube. In addition, the governing Equations of motion are obtained utilizing the Hamilton's principle and simply-simply boundary conditions are assumed. Furthermore, the Navier method is used for determining the natural frequencies of DWCNT. Hence, some parameters such as nonlocality, curvature amplitude, Winkler and Pasternak elastic foundations and length of the curved DWCNT are analyzed and discussed. The results show that, the curvature amplitude causes to increase natural frequency. However, nonlocal coefficient and elastic foundations have important role in vibration behavior of DWCNT with imperfection.

Keywords: free vibration; double walled carbon nanotubes initial imperfection; elastic foundation

1. Introduction

Carbon nanotubes (CNTs) have received remarkable attention since they were first discovered by Iijima (1991). Due to their novel electronic, mechanical and other physical and chemical properties, CNTs have potential applications in atomic-force microscopes, field emitters, nano-actuators, nano-motors, nano-bearings, nanosprings, nano-fillers for composite materials, and nanoscale electronic devices.

Extensive studies have been conducted on the mechanical properties of CNTs such as static bending analysis, Pantano *et al*. (2004), Yang and Wang (2006), free vibration analysis and dynamic response, Mitra and Gopalakrishnan (2009) and Chang and Lee (2009), buckling analysis, Wang *et al*. (2007a) and Zhang *et al*. (2006a) and postbuckling analysis, Shen and Zhang (2007) and Zhang and Shen (2007).

In recent years, the studies of CNTs using Eringen's nonlocal elasticity theory, Eringen (1983, 2002) have been attractive for scholars. The nonlocal elasticity theory has been applied to analyze the bending, buckling, and vibration behaviors of the CNTs based on the variety of beam models, Reddy and Pang (2008) and Wang *et al*. (2008) and shell models, Zhang *et al*. (2006b), Li and Kardomateas (2007). Wang (2005) discussed the molecular dispersion relationships for CNTs by

*Corresponding author, Ph.D., E-mail: jehyaei@eng.ikiu.ac.ir

http://www.techno-press.org/?journal=anr&subpage=5 ISSN: 2287-237X (Print), 2287-2388 (Online)

taking into account the small scale effect.

Wang and Hu (2005) studied flexural wave propagation in a single-walled CNT (SWNT) by using the continuum mechanics and dynamic simulation. Lu *et al*. (2007) investigated the wave propagation and vibration properties of single- or multi walled CNTs based on nonlocal beam model. Wang *et al*. (2007b) presented analytical solutions for the free vibration analysis of nonlocal Timoshenko beams. Reddy (2007) developed nonlocal theories for Euler–Bernoulli, Timoshenko, Reddy, and Levinson beams. Analytical solutions for bending, vibration and buckling were obtained considering the nonlocal effect on bending deformation, buckling load, and natural frequencies. More recently, Yang *et al*. (2008) investigated the pull-in instability of nano-switches subjected to combined electrostatic and intermolecular forces within the framework of nonlocal elasticity theory. Wang *et al*. (2006) modeled the CNTs as a nonlocal elastic cylindrical shell and dealt with the dispersion of longitudinal waves in a single-walled armchair CNT.

Wang and Varadan (2007) studied wave propagation in CNTs based on the nonlocal elastic shell theory. Hu *et al*. (2008) used nonlocal shell model to study the transverse and torsional waves in single and double-walled CNTs and verified these models by using molecular dynamic simulation. In addition, Ke *et al*. (2009) investigated the nonlinear free vibration analysis of double walled carbon nanotubes embedded in an elastic medium. In this research they used nonlocal Timoshenko beam theory. Nonlinear free vibrations of curved double walled carbon nanotubes using differential quadrature method has been studied by Cigeroglu and Samandari (2014). Mehdipour *et al*. (2012) have investigated the vibration analysis of curved single-walled carbon nanotube on a Pasternak elastic foundation. In addition, Pressure dependence of the instability of multi-walled carbon nanotubes conveying fluids was presented by He *et al*. (2008). Also, Demir and Civalek (2013) investigated torsional and longitudinal frequency and wave response of microtubules based on the nonlocal continuum and nonlocal discrete models. However, a new trigonometric beam model for buckling of strain gradient microbeams was presented by Akgöz and Civalek (2014). In addition, in another paper Civalek and Akgöz (2009), accomplished the Static analysis of single walled carbon nanotubes (SWCNT) based on Eringen's nonlocal elasticity theory, however, the mentioned papers have not considered the imperfection on vibration analysis of Double-Walled-Carbon-Nanotubes.

To the best of the authors' knowledge, there is not any research on the vibration analysis of DWCNT considering both the elastic foundation and initial imperfection in geometric based on nonlocal Timoshenko beam model, up to now. In this paper, the governing equations of motion have been obtained via the Hamilton's principle. Simply-simply supports as boundary conditions assumed. Furthermore, The Navier method has been used for determining the natural frequencies of curved DWCNT. Hence, some parameters such as nonlocality, curvature amplitude, Winkler and Pasternak elastic foundations and length of the curved DWCNT have been analyzed and discussed. The results show that, the curvature amplitude causes to increase natural frequency. However, nonlocal coefficient and elastic foundations play an important role in vibration of DWCNT with geometrical imperfection.

2. Nonlocal nanobeam model

According to Eringen's nonlocal elasticity theory, Eringen (1983, 2002), the stress at a point x in a body depends not only on the strain at point x but also on those at all other points of the body. Thus, the nonlocal stress tensor σ at point x is expressed as below, Reddy (2007)

$$\sigma = \int_v \alpha\left(\left|x' - x\right|, \tau\right) T\left(x'\right) dx' \tag{1}$$

$$T\left(x\right) = C\left(x\right) : \varepsilon\left(x\right) \tag{2}$$

where $T(x)$ is the classic, macroscopic stress tensor at point x, $\varepsilon(x)$ is the strain tensor, $C(x)$ is the fourth-order elasticity tensor and denotes the 'double-dot product'. α ($|x' - x|$, τ) is the nonlocal modulus or attenuation function incorporating into the constitutive equations the nonlocal effects at the reference point x produced by the local strain at the source x', $|x' - x|$ is the Euclidean distance, and $\tau = e_0 a/l$ is defined as small scale factor where e_0 is a constant to adjust the model to match the reliable results by experiments or other models, and a and l are the internal and external characteristic length (e.g., crack length, wavelength), respectively.

For a beam structure, the sizes in thickness and width are much smaller than the size in length. Therefore, for the beams with transverse motion in the *x–z* plane, the nonlocal constitutive relations can be approximated to one-dimensional form as below, Reddy (2007)

$$\sigma_{xx} - \left(e_0 a\right)^2 \frac{d^2 \sigma_{xx}}{dx^2} = E\,\varepsilon_{xx} \tag{3}$$

$$\sigma_{xz} - \left(e_0 a\right)^2 \frac{d^2 \sigma_{xz}}{dx^2} = E\,\varepsilon_{xz} \tag{4}$$

where E and G are Young's modulus and shear modulus, respectively, ε_{xx} is the axial strain, and γ_{xz} is the shear strain. When the nonlocal parameter $e_0 a$ is zero, we can obtain the constitutive relations of the classical theories.

3. Vibration of double walled curved nanotube

The double walled curved carbon nanotubes was modeled as a Timoshenko beam with length L, inner radius r_1, outer radius r_2, initial curvature H and equal thickness h for each tube embedded in an elastic medium. The surrounding medium is described by the Winkler foundation model with spring constant k. Based on the Timoshenko beam theory, the displacements of an arbitrary point in the beam along the x- and z-axes, defined as follows, Mehdipour *et al*. (2012)

$$\varepsilon_{xx} = \frac{\partial U_1}{\partial x} + \frac{\partial R_1}{\partial x}\frac{\partial W_1}{\partial x} + Z\frac{\partial \phi_1}{\partial x} + \frac{\partial U_2}{\partial x} + \frac{\partial R_2}{\partial x}\frac{\partial W_2}{\partial x} + Z\frac{\partial \phi_2}{\partial x} \tag{5}$$

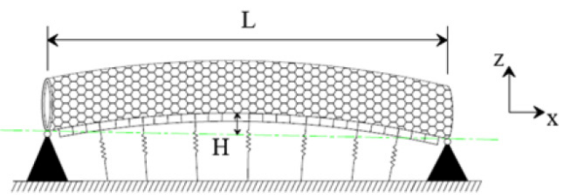

Fig. 1 Geometry of double walled carbon nanotube with an imperfection resting on elastic foundation

$$\gamma_{xz} = \frac{\partial W}{\partial x} + \phi \tag{6}$$

where R is an initial imperfection in curved carbon nanotubes. The strain energy V of the double walled curved carbon nanotubes embedded in an elastic medium can be calculated from the following equation

$$V = \frac{1}{2}\int_0^L \int_{A_1} \left(\sigma_{xx1}\varepsilon_{xx1} + \sigma_{xz1}\gamma_{xz1}\right) dA_1 dx + \frac{1}{2}\int_0^L \int_{A2} \left(\sigma_{xx2}\varepsilon_{xx2} + \sigma_{xz2}\gamma_{xz2}\right) dA_2 dx + \frac{1}{2} K \int_0^L \left(W_2\right)^2 dx \tag{7}$$

where the spring constant k is determined by the material constants of the elastic medium and A_1 and A_2 are the cross-sectional areas of the inner and outer tubes, respectively. Substituting Eqs. (5) and (6) into Eq. (7) gives

$$\begin{aligned} V = &\frac{1}{2}\int_0^L \int_{A_1} \left\{ \sigma_{xx1}\left[\frac{\partial U_1}{\partial x} + \frac{\partial R_1}{\partial x}\frac{\partial W_1}{\partial x}\right] + \sigma_{xx1} Z \frac{\partial \phi_1}{\partial x} + \sigma_{xz1}\gamma_{xz1} \right\} dA_1 dx + \\ &\frac{1}{2}\int_0^L \int_{A_2} \left\{ \sigma_{xx2}\left[\frac{\partial U_2}{\partial x} + \frac{\partial R_2}{\partial x}\frac{\partial W_2}{\partial x}\right] + \sigma_{xx2} Z \frac{\partial \phi_2}{\partial x} + \sigma_{xz2}\gamma_{xz2} \right\} dA_2 dx + \frac{1}{2} K \int_0^L \left(W_2\right)^2 dx \end{aligned} \tag{8}$$

Subscript i = 1 and 2 in U_i, ψ_i, W_i, σ_{xxi}, σ_{xzi}, ε_{xxi}, γ_{xzi}, N_{xi}, M_{xi}, Q_{xi} refers to the inner and outer tubes respectively. The normal resultant force N_{xi}, bending moment M_{xi}, and transverse shear force Q_{xi} are defined as

$$N_{xi} = \int_{Ai} \sigma_{xxi} dA_i \tag{9a}$$

$$M_{xi} = \int_{Ai} \sigma_{xxi} z dA_i \tag{9b}$$

$$Q_{xi} = \int_{Ai} \sigma_{xzi} dA_i \tag{9c}$$

The work done by the *vdW* forces is denoted by p_q, Mehdipour *et al*. (2012)

$$P_q = \frac{1}{2}\int_0^L q_1 W_1 dx + \frac{1}{2}\int_0^L q_2 W_2 dx \tag{10}$$

For the double walled curved carbon nanotubes, the *vdW* interaction forces between the two tubes can be expressed as

$$q_1 = C_1\left(W_2 - W_1\right) \tag{11a}$$

$$q_2 = -C_1\left(W_2 - W_1\right) \tag{11b}$$

where q_1 and q_2 are van der Waals forces and C_1 is the *vdW* interaction coefficient as below

$$C_1 = \frac{320 \times \left(2r_1\right)^{erg}\!/_{cm^2}}{0.16\Delta^2} \tag{12a}$$

$$\Delta = 0.142nm \tag{12b}$$

The kinetic energy T is given by

$$\begin{aligned} T = & \frac{1}{2}\int_0^L \left[\rho A_1 \left(\frac{\partial U_1}{\partial t} \right)^2 + \rho A_1 \left(\frac{\partial W_1}{\partial t} \right)^2 + \rho I_1 \left(\frac{\partial \varphi_1}{\partial t} \right)^2 \right] dx + \\ & \frac{1}{2}\int_0^L \left[\rho A_2 \left(\frac{\partial U_2}{\partial t} \right)^2 + \rho A_2 \left(\frac{\partial W_2}{\partial t} \right)^2 + \rho I_2 \left(\frac{\partial \varphi_2}{\partial t} \right)^2 \right] dx \end{aligned} \tag{13}$$

where I_1 and I_2 are the second moments of area of the inner and outer tubes and ρ is the mass density of the CNTs. The equations of motion of the nonlocal double walled curved carbon nanotubes embedded in an elastic medium can be derived from the Hamilton's principle

$$\int_0^t \left(\delta T - \delta V + \delta P_q \right) dt = 0 \tag{14}$$

Substituting Eqs. (8), (10), and (13) into Eq. (14), integrating by parts and setting the coefficients of δU_i, δW_i and $\delta \psi_i$ to zero lead to the equations of motion as follows

$$\frac{\partial N_{xi}}{\partial x} = \rho A_i \frac{\partial^2 U_i}{\partial t^2} \tag{15a}$$

$$\frac{\partial Q_{xi}}{\partial x} + \frac{\partial}{\partial x}\left(N_{xi} \frac{\partial W_i}{\partial x} \right) + \frac{\partial}{\partial x}\left(N_{xi} \frac{\partial R}{\partial x} \right) + q_i - m_0 K W_2 = \rho A_i \frac{\partial^2 W_i}{\partial t^2} \tag{15b}$$

$$\frac{\partial M_{xi}}{\partial x} - Q_{xi} = \rho I_i \frac{\partial^2 \phi_i}{\partial t^2} \tag{15c}$$

where $m_0 = 0$ for $i = 1$ and $m_0 = 1$ for $i = 2$.

Note that expressions of the normal resultant force, bending moment and shear force in the nonlocal beam theory are different from those in the classical Timoshenko beam theory due to the nonlocal constitutive relations (3) and (4). From Eqs. (3), (4), (5), (6) and (15), the normal resultant force, bending moment and shear force can be expressed as

$$N_{xi} - \left(e_0 a \right)^2 \frac{\partial^2 N_{xi}}{\partial x^2} = EA_i \left(\frac{\partial U_i}{\partial x} + \frac{\partial R_i}{\partial x} \frac{\partial W_i}{\partial x} \right) \tag{16a}$$

$$M_{xi} - \left(e_0 a \right)^2 \frac{\partial^2 M_{xi}}{\partial x^2} = EI_i \frac{\partial \varphi}{\partial x} \tag{16b}$$

$$Q_{xi} - \left(e_0 a \right)^2 \frac{\partial^2 Q_{xi}}{\partial x^2} = k_{si} GA_i \left(\frac{\partial W_i}{\partial x} + \phi_i \right) \tag{16c}$$

where K_{si} (i = 1, 2) is the shear correction factor depending on the shape of the cross-section of the tubes. By substituting Eq. (16) into Eq. (15), the explicit expressions of nonlocal normal resultant force N_{xi}, bending moment M_{xi} and shear force Q_{xi} can be obtained as

$$N_{xi} = EA_i\left(\frac{\partial U_i}{\partial x} + \frac{\partial R_i}{\partial x}\frac{\partial W_i}{\partial x}\right) + \left(e_0 a\right)^2 \rho A_i \frac{\partial^3 U_i}{\partial x \partial t^2} \tag{17a}$$

$$\begin{aligned} M_{xi} &= EI_i \frac{\partial \varphi_i}{\partial x} \\ &+\left(e_0 a\right)^2\left[\rho I_i \frac{\partial^3 \varphi_i}{\partial x \partial t^2} + \rho A_i \frac{\partial^2 W_i}{\partial t^2} - \frac{\partial}{\partial x}\left(N_{xi}\frac{\partial W_i}{\partial x}\right) - \frac{\partial}{\partial x}\left(N_{xi}\frac{\partial R_i}{\partial x}\right) - q_i + m_0 K W_2\right] \end{aligned} \tag{17b}$$

$$\begin{aligned} Q_{xi} &= k_{si} G A_i\left(\frac{\partial W_i}{\partial x} + \varphi_i\right) \\ &+\left(e_0 a\right)^2\left[\rho A_i \frac{\partial^3 W_i}{\partial x \partial t^2} - \frac{\partial^2}{\partial x^2}\left(N_{xi}\frac{\partial W_i}{\partial x}\right) - \frac{\partial^2}{\partial x^2}\left(N_{xi}\frac{\partial R_i}{\partial x}\right) - \frac{\partial q_i}{\partial x} + m_0 K \frac{\partial W_2}{\partial x}\right] \end{aligned} \tag{17c}$$

Then, the equations of motion for the nonlocal double walled curved carbon nanotubes can be derived by inserting Eq. (17) into Eq. (15)

$$EA_i\left(\frac{\partial^2 U_i}{\partial x^2} + \frac{\partial^2 R_i}{\partial x^2}\frac{\partial W_i}{\partial x} + \frac{\partial^2 W_i}{\partial x^2}\frac{\partial R_i}{\partial x}\right) = \rho A_i \frac{\partial^2}{\partial t^2}\left(U_i - \left(e_0 a\right)^2 \frac{\partial^2 U_i}{\partial x^2}\right) \tag{18a}$$

$$\begin{aligned} & k_{si} G A_i\left(\frac{\partial^2 W_i}{\partial x^2} + \frac{\partial \varphi_i}{\partial x}\right) + \\ & \left(e_0 a\right)^2 \left(\begin{array}{l} \rho A_i \frac{\partial^4 W_i}{\partial x^2 \partial t^2} + \rho A_i K_w \frac{\partial^6 W_i}{\partial x^4 \partial t^2} - K_G \frac{\partial^8 W_i}{\partial x^6 \partial t^2} \\ -\frac{\partial^3}{\partial x^3}\left[EA_i\left(\frac{\partial U_i}{\partial x}\frac{\partial W_i}{\partial x} + \frac{\partial R_i}{\partial x}\left(\frac{\partial W_i}{\partial x}\right)^2\right) + \left(e_0 a\right)^2 \rho A_i \frac{\partial^3 U_i}{\partial x \partial t^2}\frac{\partial W_i}{\partial x}\right] \end{array}\right) \\ & -\frac{\partial^3}{\partial x^3}\left[EA_i\left(\frac{\partial U_i}{\partial x}\frac{\partial R_i}{\partial x} + \left(\frac{\partial R_i}{\partial x}\right)^2\frac{\partial W_i}{\partial x}\right) + \left(e_0 a\right)^2 \rho A_i \frac{\partial^3 U_i}{\partial x \partial t^2}\frac{\partial R_i}{\partial x}\right] \\ & -\frac{\partial^2 q_i}{\partial x^2} + m_0 K \frac{\partial^2 W_2}{\partial x^2} + \frac{\partial}{\partial x}\left[EA_i\left(\frac{\partial U_i}{\partial x}\frac{\partial W_i}{\partial x} + \frac{\partial R_i}{\partial x}\left(\frac{\partial W_i}{\partial x}\right)^2\right) + \left(e_0 a\right)^2 \rho A_i \frac{\partial^3 U_i}{\partial x \partial t^2}\frac{\partial W_i}{\partial x}\right] \\ & +\frac{\partial}{\partial x}\left[EA_i\left(\frac{\partial U_i}{\partial x}\frac{\partial R_i}{\partial x} + \left(\frac{\partial R_i}{\partial x}\right)^2\frac{\partial W_i}{\partial x}\right) + \left(e_0 a\right)^2 \rho A_i \frac{\partial^3 U_i}{\partial x \partial t^2}\frac{\partial R_i}{\partial x}\right] + q_i - m_0 K W_2 - \rho A_i \frac{\partial^2 W_i}{\partial t^2} \end{aligned} \tag{18b}$$

$$EI_i \frac{\partial^2 \varphi}{\partial x^2} - k_{si} G A_i\left(\frac{\partial W_i}{\partial x} + \varphi_i\right) = \rho I_i \frac{\partial^2}{\partial t^2}\left(\varphi_i - \left(e_0 a\right)^2 \frac{\partial^2 \varphi_i}{\partial x^2}\right) \tag{18c}$$

4. Solution method

In this study, the free vibration equation of the curved DWCNT has been investigated by using the nonlocal TBM. The influences of transverse shear deformation and rotary inertia on the vibration frequencies are also investigated.

The Navier method is a powerful solution technique to solve the differential equations. The Navier method of decomposition is used to obtain the governing ordinary differential equation (ODE) from a partial counterpart. For one-term approximation the deflection of the beam U_i (x, t), W_i (x, t), φ_i (x, t) separates as

$$U_1 = \sum_{m=1}^{\infty} U_1 Cos\left(\frac{m\pi}{l}x\right)e^{i\omega_n t} \tag{19a}$$

$$W_1 = \sum_{m=1}^{\infty} W_1 Sin\left(\frac{m\pi}{l}x\right)e^{i\omega_n t} \tag{19b}$$

$$\varphi_1 = \sum_{m=1}^{\infty} \varphi_1 Cos\left(\frac{m\pi}{l}x\right)e^{i\omega_n t} \tag{19c}$$

$$U_2 = \sum_{m=1}^{\infty} U_2 Cos\left(\frac{m\pi}{l}x\right)e^{i\omega_n t} \tag{19d}$$

$$W_2 = \sum_{m=1}^{\infty} U_2 Sin\left(\frac{m\pi}{l}x\right)e^{i\omega_n t} \tag{19e}$$

$$\varphi_2 = \sum_{m=1}^{\infty} \varphi_2 Cos\left(\frac{m\pi}{l}x\right)e^{i\omega_n t} \tag{19f}$$

By inserting Eq. (19) into governing equation, natural frequencies can be obtained

$$\left[\omega^2 \rho A_1\left[1+(e_0 a)^2\left(\frac{m\pi}{l}\right)^2\right] - EA_1\left(\frac{m\pi}{l}\right)^2\right]U_1 = 0 \tag{20a}$$

$$\begin{aligned}
&\left[-K_{s1}GA_1\left(\frac{m\pi}{l}\right)^2 - C_1 - C_1(e_0 a)^2\left(\frac{m\pi}{l}\right)^2 + H\left(\frac{m\pi}{l}\right)^2 + \rho A_1\left[\omega^2 - \omega^2(e_0 a)^2\left(\frac{m\pi}{l}\right)^2\right]\right]W_1 \\
&-\left[K_{s1}GA_1\left(\frac{m\pi}{l}\right)\right]\varphi_1 + \left[C_1 + C_1(e_0 a)^2\left(\frac{m\pi}{l}\right)^2 - H\left(\frac{m\pi}{l}\right)^2 - m_0 k\left(1+(e_0 a)^2\left(\frac{m\pi}{l}\right)^2\right)\right]W_2 \\
&+\left[\rho A_1\left[(e_0 a)^2\omega^2\left(\frac{m\pi}{l}\right)^2 + (e_0 a)^2\omega^2\left(\frac{m\pi}{l}\right)^4\right]\right]U_1 = 0
\end{aligned} \tag{20b}$$

$$\left[-EI_1\left(\frac{m\pi}{l}\right)^2 - K_{s1}GA_1 + \rho I_1\left(\omega^2 + \omega^2\left(\frac{m\pi}{l}\right)^2\right)\right]\varphi_1 - K_{s1}GA_1\left(\frac{m\pi}{l}\right)W_1 = 0 \tag{20c}$$

$$\left[\omega^2\rho A_2\left[1+\left(e_0 a\right)^2\left(\frac{m\pi}{l}\right)^2\right] - EA_2\left(\frac{m\pi}{l}\right)^2\right]U_2 = 0 \tag{20d}$$

$$\begin{aligned}&\begin{bmatrix}-K_{s2}GA_2\left(\frac{m\pi}{l}\right)^2 - C_1 - C_1\left(e_0 a\right)^2\left(\frac{m\pi}{l}\right)^2 + H\left(\frac{m\pi}{l}\right)^2 \\ -m_0 k\left(1+\left(e_0 a\right)^2\left(\frac{m\pi}{l}\right)^2\right) + \rho A_2\left[\omega^2 - \omega^2\left(e_0 a\right)^2\left(\frac{m\pi}{l}\right)^2\right]\end{bmatrix}W_2 \\ &-\left[K_{s2}GA_2\left(\frac{m\pi}{l}\right)\right]\varphi_2 + \left[C_1 + C_1\left(e_0 a\right)^2\left(\frac{m\pi}{l}\right)^2 + H\left(\frac{m\pi}{l}\right)^2\right]W_1 \\ &+\left[\rho A_2\left[\left(e_0 a\right)^2\omega^2\left(\frac{m\pi}{l}\right)^2 + \left(e_0 a\right)^2\omega^2\left(\frac{m\pi}{l}\right)^4\right]\right]U_2 = 0\end{aligned} \tag{20e}$$

$$\left[-EI_2\left(\frac{m\pi}{l}\right)^2 - K_{s2}GA_2 + \rho I_2\left(\omega^2 + \omega^2\left(\frac{m\pi}{l}\right)^2\right)\right]\varphi_2 - K_{s2}GA_2\left(\frac{m\pi}{l}\right)W_2 = 0 \tag{20f}$$

5. Numerical results

In this case study, the diameter, aspect ratio, thickness of DWCNT and Young's modulus of the nanotube are assumed to be d_e = 3.19 nm, t_c = 0.137 nm, and E = 2.407 TPa, respectively. The mass density of the DWCNT is 2300 kg/m^3 with nonlocal parameter e_0a of 2 nm. Also, the Winkler modulus and Pasternak modulus are estimated at the values of K_w = 1 MPa and K_G = 5 nN, in that order, Murmu and Pradhan (2009). Moreover, the amplitude of curve H is 1 nm, Mayoof and Hawwa (2009).

To validate the results the amplitude of curvature assumes to be zero and elastic foundation is eliminated. One can find the comparison results in Table 1.

As it is shown in Table 1, present work has good consistency with the corresponding results reported by Ke *et al*. (2009).

Table 1 Comparison of first four natural frequency with Ke *et al*. (2009)

Mode	Ke *et al*. (2009)	Present	Error
1	3.1438	3.1416	0.07%
2	6.2832	6.2832	0%
3	9.3509	9.4247	0.78%
4	12.536	12.5664	0.13%

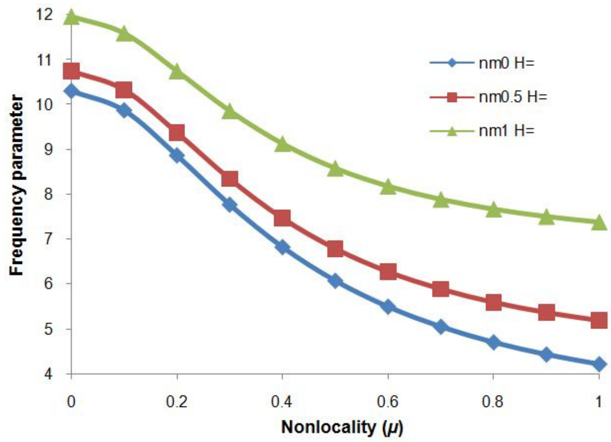

Fig. 2 Variation frequency parameter respect to nonlocality with various curvatures

A variation of the frequency parameter of curved DWCNT with respect to nonlocal parameters is depicted in Fig. 2. It can be seen from the figure that the frequency parameter (Ω) decreases with the increase of values of the dimensionless nonlocal parameter (μ). Decreasing the dimensionless natural frequency against increasing nonlocal parameter occurs for all the three cases considered for amplitude of curvatures. Hence the size effects in nonlocal elastic model reflected in the vibration of the curved DWCNT.

In addition, the effect of Winkler and Pasternak elastic foundations are discussed in this research. For this purpose, the variation of frequency parameter with respect to amplitude of curvatures considering the various Winkler and Pasternak values, is plotted in Figs. 3 and 4.

As it is shown in Figs. 3 and 4, by increasing the amplitude of curvature, frequency parameter increases. Also it should be noted that, increasing the values of Winkler elastic foundation, leads to decreasing the natural frequency. It is interesting to say that the Pasternak foundation has inverse effect on vibration of curved DWCNT.

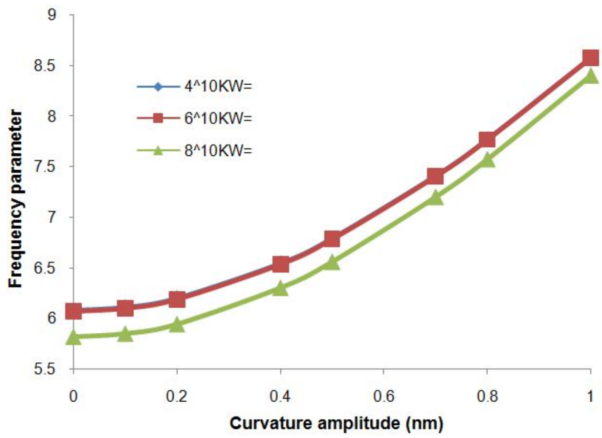

Fig. 3 Variation frequency parameter respect to amplitude of curvature with different Winkler elastic foundations

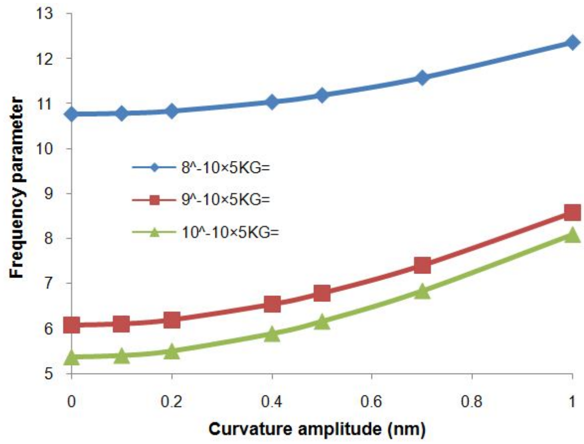

Fig. 4 Variation frequency parameter respect to amplitude of curvature with different Pasternak elastic foundations

Moreover, to see the effects of curvature clearly, the difference percent (DP) is defined as a parameter that shows the percent increment of frequency for a curved DWCNT ($H \neq 0$ nm) compared with a straight nanotube ($H = 0$ nm).

$$DifferencePercent\left(DP\right)=\frac{\left(\Omega_{TBM}^{H=1nm}-\Omega_{TBM}^{H=0nm}\right)\times 100}{\Omega_{TBM}^{H=0nm}}$$

Certainly, DP gives a better illustration for the pure effects of the amplitude of curvature H. Figs. 5, 6, 7 and 8, represent the difference percent DP as a function of the waviness amplitude H, while the effects of a certain parameter such as the stiffness of model, the length of curved DWCNT and the nonlocal parameter have been evaluated in each figure. Obviously, the variation of fundamental frequencies is increased when the waviness is amplified, in all the figures.

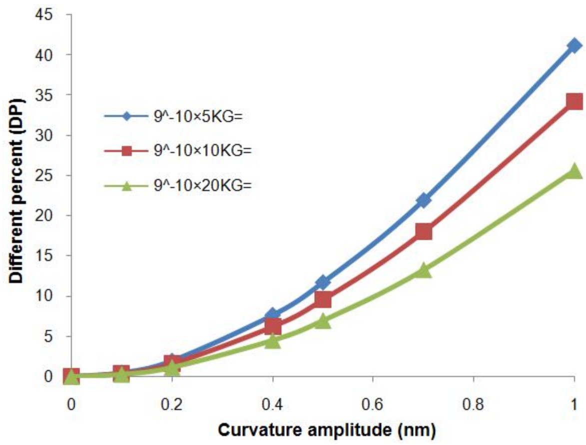

Fig. 5 The different percent DP against the curvature amplitude with different values of Pasternak elastic foundation

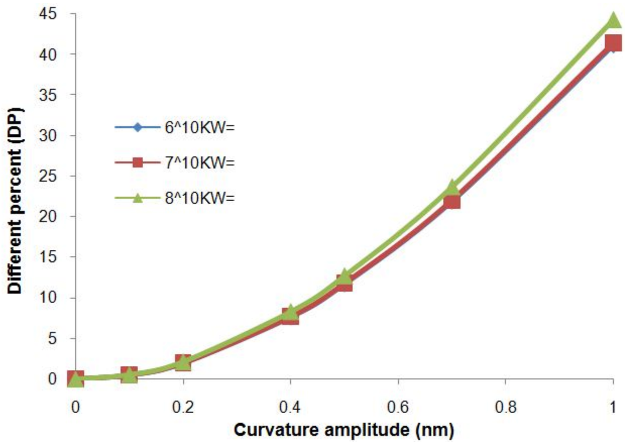

Fig. 6 The different percent DP against the curvature amplitude with different values of Winkler elastic foundation

As it is shown in Figs. 5 and 6, by increasing the amplitude of curvature, difference percent increases. It means, in higher curvature amplitudes the variation of curvature amplitude becomes sensitive in frequency parameter. Also it should be noted that, by increasing the values of Winkler and Pasternak elastic foundation, difference percent increases. It is interesting to say that Pasternak foundation has a more important role than Winkler foundation in difference percent.

In addition, to see the effect of nonlocality a length of the curved DWCNT on difference percent, the variation of difference percent with respect to amplitude of curvatures with various nonlocality and length of the curved DWCNT, are plotted in Figs. 7 and 8.

As it is shown in Figs. 7 and 8, by increasing amplitude of curvature, difference percent increases. It means, in higher curvature amplitude the variation of curvature amplitude becomes sensitive in frequency parameter. Also it should be noted that, by increasing the values of nonlocality, difference percent increases. It is interesting to say that, by increasing the length of the curved DWCNT, difference percent decreases.

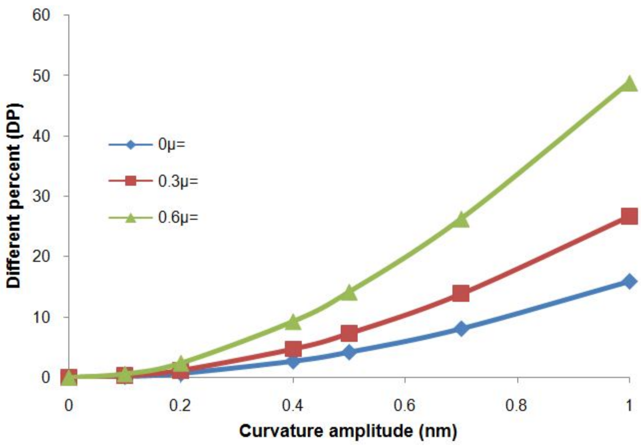

Fig. 7 The different percent DP against the curvature amplitude with different values of nonlocality

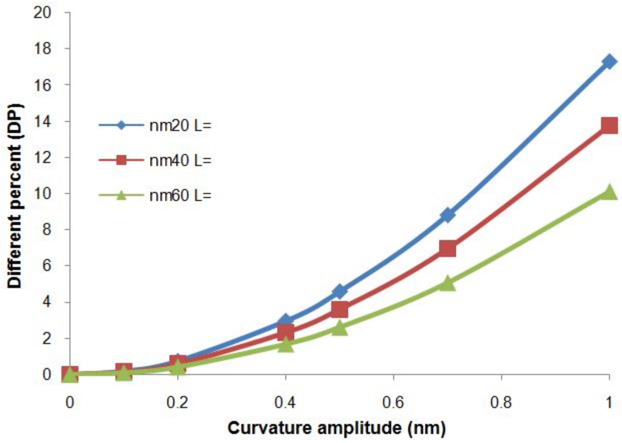

Fig. 8 The different percent DP against the curvature amplitude with different values of the length

Table 2 Natural frequencies respect to various parameter

H	μ	$K_W = 10^6$			$K_W = 10^8$		
		$K_G = 5\times10^{-9}$	$K_G = 10\times10^{-9}$	$K_G = 20\times10^{-9}$	$K_G = 5\times10^{-9}$	$K_G = 10\times10^{-9}$	$K_G = 20\times10^{-9}$
$H = 0$ mn	$\mu = 0$	10.3096	10.7283	11.5200	10.6469	11.0528	11.8228
	$\mu = 0.1$	9.8755	10.3118	11.1332	10.1648	10.5892	11.3906
	$\mu = 0.2$	8.8700	9.3533	10.2518	9.0412	9.5158	10.4003
	$\mu = 0.3$	7.7716	8.3190	9.3178	7.7985	8.3441	9.3402
	$\mu = 0.4$	6.8235	7.4409	8.5430	6.7062	7.3335	8.4496
	$\mu = 0.5$	6.0721	6.7585	7.9557	5.8206	6.5335	7.7654
	$\mu = 0.6$	5.4920	6.2425	7.5223	5.1185	5.9166	7.2541
	$\mu = 0.7$	5.0449	5.8531	7.2024	4.5613	5.4418	6.8723
	$\mu = 0.8$	4.6978	5.5567	6.9636	4.1150	5.0736	6.5845
	$\mu = 0.9$	4.4252	5.3282	6.7826	3.7531	4.7847	6.3646
	$\mu = 1$	4.2084	5.1496	6.6433	3.4560	4.5553	6.1941
$H = 1$ mn	$\mu = 0$	11.9569	12.3197	13.0150	12.2489	12.6033	13.2838
	$\mu = 0.1$	11.5847	11.9588	12.6739	11.8323	12.1988	12.9006
	$\mu = 0.2$	10.7404	11.1429	11.9071	10.8823	11.2797	12.0352
	$\mu = 0.3$	9.8528	10.2901	11.1131	9.8740	10.3104	11.1319
	$\mu = 0.4$	9.1236	9.5941	10.4720	9.0362	9.5111	10.3959
	$\mu = 0.5$	8.5762	9.0751	9.9987	8.4000	8.9088	9.8479
	$\mu = 0.6$	8.1757	8.6977	9.6574	7.9296	8.4667	9.4500
	$\mu = 0.7$	7.8824	8.4225	9.4103	7.5819	8.1421	9.1602
	$\mu = 0.8$	7.6648	8.2193	9.2290	7.3221	7.9007	8.9463
	$\mu = 0.9$	7.5008	8.0666	9.0931	7.1250	7.7184	8.7857
	$\mu = 1$	7.3750	7.9498	8.9897	6.9731	7.5784	8.6629

According to Table 2, it can be seen obviously, that the dimensionless frequency increases by increasing the amplitude of curvatures. It is interesting to say that natural frequencies also increase by increase the Pasternak elastic foundation. However, dimensionless natural frequencies also decrease by increasing the nonlocal coefficient. The results in Table 2 can be used for design of curved DWCNT in the future.

6. Conclusions

Free vibration analysis of double walled carbon nanotubes (DWCNT) embedded in elastic medium with initial imperfection is investigated in this study. In this paper, Timoshenko beam theory is employed. However the nonlocal theory is used for modeling the nano scale of nanotube. In addition, the governing equations were obtained due to the Hamilton principle. Simply-simply boundary conditions are assumed for this case. Furthermore, the Navier exact solution was used for determining the natural frequencies of DWCNT with imperfection. Hence, some parameter such as nonlocality, curvature amplitude, Winkler and Pasternak elastic foundations and length of the DWCNT were discussed. According to the results, the curvature amplitude causes to increase the natural frequency. However, nonlocal coefficient and elastic foundations play an important role in vibration behavior of imperfect DWCNT.

References

Akgöz, B. and Civalek, Ö. (2014), "A new trigonometric beam model for buckling of strain gradientmicrobeams", *Int. J. Mech. Sci.*, **81**, 88-94.

Chang, W.J. and Lee, H.L. (2009), "Free vibration of a single-walled carbon nanotube containing a fluid flow using the Timoshenko beam model", *Phys. Lett. A*, **373**(10), 982-985.

Cigeroglu, E. and Samandari, H. (2014), "Nonlinear free Vibrations of curved double walled carbon Nanotubes using differential quadrature method", *Physica E: Low-dimensional Systems and Nanostructures*, **64**, 95-105.

Civalek, Ö, and Akgöz, B. (2009), "Static analysis of single walled carbon nanotubes (SWCNT) based on Eringen's nonlocal elasticity theory", *Int. J. Eng. Appl. Sci.*, **1**(2), 47-56.

Demir, C. and Civalek, Ö. (2013), "Torsional and longitudinal frequency and wave response of microtubules based on the nonlocal continuum and nonlocal discrete models", *Appl. Math. Model.*, **37**(22), 9355-9367.

Eringen, A.C. (1983), "On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves", *J. Appl. Phys.*, **54**(9), 4703-4710.

Eringen, A.C. (2002), *Nonlocal Continuum Field Theories*, Springer Science & Business Media.

He, X.Q., Wang, C.M., Yan, Y., Zhang, L.X. and Nie, G.H. (2008), "Pressure dependence of the instability of multiwalled carbon nanotubes conveying fluids", *Arch. Appl. Mech.*, **78**(8), 637-648.

Hu, Y.G., Liew, K.M., Wang, Q., He, X.Q. and Yakobson, B.I. (2008), "Nonlocal shell model for elastic wave propagation in single-and double-walled carbon nanotubes", *J. Mech. Phys. Solids*, **56**(12), 3475-3485.

Iijima, S. (1991), "Helical microtubules of graphitic carbon", *Nature*, **354**(6348), 56-58.

Ke, L.L., Xiang, Y., Yang, J. and Kitipornchai, S. (2009), "Nonlinear free vibration of embedded double-walled carbon nanotubes based on nonlocal Timoshenko beam theory", *Computat. Mater. Sci.*, **47**(2), 409-417.

Li, R. and Kardomateas, G.A. (2007), "Thermal buckling of multi-walled carbon nanotubes by nonlocal elasticity", *J. Appl. Mech.*, **74**(3), 399-405.

Lu, P., Lee, H.P., Lu, C. and Zhang, P.Q. (2007), "Application of nonlocal beam models for carbon

nanotubes", *Int. J. Solid. Struct.*, **44**(16), 5289-5300.
Mayoof, F.N. and Hawwa, M.A. (2009), "Chaotic behavior of a curved carbon nanotube under harmonic excitation", *Chaos, Solitons & Fractals*, **42**(3), 1860-1867.
Mehdipour, I., Barari, A., Kimiaeifar, A. and Domairry, G. (2012), "Vibrational analysis of curved single-walled carbon nanotube on a Pasternak elastic foundation", *Adv. Eng. Software*, **48**, 1-5.
Mir, M., Hosseini, A. and Majzoobi, G.H., (2008), "A numerical study of vibrational properties of single-walled carbon nanotubes", *Computat. Mater. Sci.*, **43**(3), 540-548.
Mitra, M. and Gopalakrishnan, S. (2009), "Wave propagation in multi-walled carbon nanotube", *Computat. Mater. Sci.*, **45**(2), 411-418.
Murmu, T. and Pradhan, S.C. (2009), "Buckling analysis of a single-walled carbon nanotube embedded in an elastic medium based on nonlocal elasticity and Timoshenko beam theory and using DQM", *Physica E: Low-dimensional Systems and Nanostructures*, **41**(7), 1232-1239.
Pantano, A., Parks, D.M. and Boyce, M.C. (2004), "Mechanics of deformation of single-and multi-wall carbon nanotubes", *J. Mech. Phys. Solids*, **52**(4), 789-821.
Reddy, J.N. (2007), "Nonlocal theories for bending, buckling and vibration of beams", *Int. J. Eng. Sci.*, **45**(2), 288-307.
Reddy, J.N. and Pang, S.D. (2008), "Nonlocal continuum theories of beams for the analysis of carbon nanotubes", *J. Appl. Phys.*, **103**(2), 023511.
Shen, H.S. and Zhang, C.L. (2007), "Postbuckling of double-walled carbon nanotubes with temperature dependent properties and initial defects under combined axial and radial mechanical loads", *Int. J. Solid. Struct.*, **44**(5), 1461-1487.
Wang, Q, (2005), "Wave propagation in carbon nanotubes via nonlocal continuum mechanics", *J. Appl. Phys.*, **98**(12), 124301.
Wang, L. (2009), "Dynamical behaviors of double-walled carbon nanotubes conveying fluid accounting for the role of small length scale", *Computat. Mater. Sci.*, **45**(2), 584-588.
Wang, L. and Hu, H. (2005), "Flexural wave propagation in single-walled carbon nanotubes", *Phys. Rev. B*, **71**(19), 195412.
Wang, Q. and Varadan, V.K. (2007), "Application of nonlocal elastic shell theory in wave propagation analysis of carbon nanotubes", *Smart Mater. Struct.*, **16**(1), 178.
Wang, L., Hu, H. and Guo, W. (2006), "Validation of the non-local elastic shell model for studying longitudinal waves in single-walled carbon nanotubes", *Nanotechnology*, **17**(5), 1408.
Wang, X., Lu, G. and Lu, Y.J. (2007a), "Buckling of embedded multi-walled carbon nanotubes under combined torsion and axial loading", *Int. J. Solid. Struct.*, **44**(1), 336-351.
Wang, C.M., Zhang, Y.Y. and He, X.Q. (2007b), "Vibration of nonlocal Timoshenko beams", *Nanotechnology*, **18**(10), 105401.
Wang, C.M., Kitipornchai, S., Lim, C.W. and Eisenberger, M. (2008), "Beam bending solutions based on nonlocal Timoshenko beam theory", *J. Eng. Mech.*, **134**(6), 475-481.
Yang, H.K. and Wang, X. (2006), "Bending stability of multi-wall carbon nanotubes embedded in an elastic medium", *Model. Simul. Mater. Sci. Eng.*, **14**(1), p. 99.
Yang, J., Jia, X.L. and Kitipornchai, S. (2008), "Pull-in instability of nano-switches using nonlocal elasticity theory", *J. Phys. D: Appl. Phys.*, **41**(3), 035103.
Zhang, C.L. and Shen, H.S. (2007), "Buckling and postbuckling of single-walled carbon nanotubes under combined axial compression and torsion in thermal environments", *Phys. Rev. B*, **75**(4), 045408.
Zhang, Y.Y., Wang, C.M. and Tan, V.B.C. (2006a), "Buckling of multiwalled carbon nanotubes using Timoshenko beam theory", *J. Eng. Mech.*, **132**(9), 952-958.
Zhang, Y.Y., Tan, V.B.C. and Wang, C.M. (2006b), "Effect of chirality on buckling behavior of single-walled carbon nanotubes", *J. Appl. Phys.*, **100**(7), 074304.

CC